Measurement and Measurement Uncertainty

Measurement

- To measure means quantifying a physical quantity by a process whose result is called measurement.

- The measurement **must be repeated also by others** and therefore must be communicated unequivocally. Therefore, with the measurement it is necessary to provide at least **the value**, **the uncertainty** and **the measurement unit**

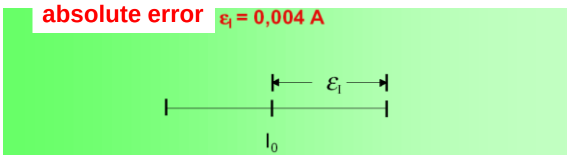
Value, Uncertainty and Measurement Unit

- The value - is the result of the comparison between the measurand and a reference quantity (STANDARD).
- The uncertainty of the measurement - is the degree of dispersion of the values attributed to the measurand during different measurements and is, therefore, indicative of the value (and also of the cost) of the measurement instrumentation.
- The measurement unit - must be internationally recognized for the purpose of better communication of the result.

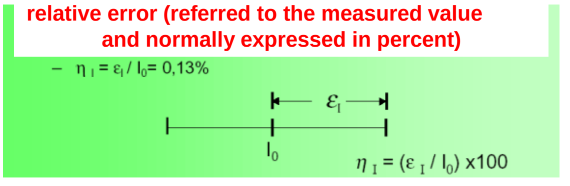
Measurement Examples

- Open voltage of a battery (9.6 ± 0.2) V
- Resistance of a resistor (12.5 ± 0.1) Ω
- We communicate the value, the uncertainty, and the measurement unit.

Absolute and Relative Errors



$I = I_o \pm \varepsilon_I$



$I = I_o \pm \eta_I\%$

Link between dB -> %

$$a_x(dB) = 20\log_{10}\left(1 + \frac{a_x(\%)}{100}\right)$$

$$\frac{a_x(dB)}{20} = \log_{10}\left(1 + \frac{a_x(\%)}{100}\right)$$

$$10^{\frac{a_x(dB)}{20}} = 1 + \frac{a_x(\%)}{100}$$

$$a_x(\%) = 100\left(10^{\frac{a_x(dB)}{20}} - 1\right)$$

How to Write the Result of a Measurement

- In order to correctly write the result of a measurement it is necessary to make some considerations:

- **Fractional part** of a number are the digits after the decimal separator (eg 7.543624 -> fractional part = 543624)

- **Significant digits** of a number are the digits after the zeros (eg 0.00254 significant digits = 254)

- When making a measurement, the value read on the instrument is initially reported with all its decimal digits (**eg 7.543624**).

- Subsequently the uncertainty is evaluated with all its decimal digits (**eg 0.00254**).

- The uncertainty is written considering at most two significant digits rounding to the upper value (**eg 0.0026**).

- Write the value with the same number of digits in the fractional part of uncertainty rounding to the nearest (**eg 7.5436**)

- Write the measurement result:

$$\mathbf{m = 7.5436 \pm 0.0026}$$

Errors and Uncertainties (random effects)

- In the measurement process there are many **factors of influence**: temperature, humidity, vibrations or electrical and electromagnetic disturbances of the environment in which the measurement takes place, etc.
- All these factors (influence quantities) interact in various ways in the measurement process, so if this is repeated different results are obtained, resulting in a **dispersion of the measured values**.
- These cited factors act randomly in the measurement process therefore if the measurement is repeated N times (with $N \rightarrow \infty$) and an average is operated, their effect tends to cancel itself (**random effects**).

Errors and Uncertainties (systematic effects)

- The measurement is also influenced by **the non-ideal behaviors of the various elements of the measurement system** (defects in the models and in the standards) that give rise to errors always in the same direction (**systematic effects**) (they cannot be removed with an average process).
- With reference to systematic effects, in some cases it is possible to estimate the entity and the sign of the error and therefore it is possible to correct the measurement (**calibration**) - in this case we say "errors" (systematic errors) . However, even when calibration is possible, residual errors always remain due to the non-ideality of the calibration standards used.

Type A Assessment

- If the measurement is supposed to be affected by random uncertainties, the uncertainty can be evaluated with statistical methods. **These methods are said to be of type A**. If the measurement is repeated under the same conditions for many times and the histogram is traced, we see that this tends to a Gaussian (Central limit theorem)

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x - x_m)^2}{2\sigma^2} \right]$$

- with x_m mean value and σ^2 variance.
- the probability that a value taken at random from the measurements made falls in the range $x_m \pm \sigma$ is 68.4%.

Estimates

- Statistical theory shows that, for any p(x), **the best estimate of the mean value** (x_m) is given by the experimental average m_N obtained on N independent observations x_k as:

$$m_N = \frac{1}{N}\sum_{k=1}^{N} x_k$$

- **The best estimate of the variance of the mean** is given by :

$$\sigma_M^2 = \frac{1}{N(N-1)}\sum_{k=1}^{N} (m_k - m_N)^2$$

- Therefore, if we want to quantify the uncertainty we assume σ_M as an experimental estimate of the uncertainty.
- σ_M is the type A standard uncertainty and is indicated by the letter u.
- As is known, the probability that a value taken at random falls in the interval $\pm\,\sigma$ is 68.4%. If the probability of 68.4% is not considered sufficient, this value can be increased by introducing the **extended uncertainty K $\times$ u where K is called coverage factor**.
- With K = 2 there is a probability of 95.4% (2u). With K = 3 there is a probability of 99.7% (3u). Then the measurement can be expressed as:

$$x = m_N \pm Ku$$

Type B Assessments

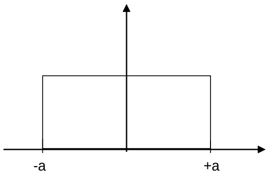
- In this class, all the evaluations of uncertainties that are not obtained through the standard deviation of repeated measurements (as for that of type A) are included.

Example Procedure for Type B Assessments

- An estimate is made of the limits of the variations on the measurement caused by a source of uncertainty, ie the maximum deviation is evaluated.
- Subsequently, a certain probability distribution is assumed between these limits.
- Finally, we calculate an equivalent standard deviation that represents the standard type B uncertainty.

Rectangular Distribution

- The rectangular distribution is used when the variation limits are known and it can be assumed that all the values are equiprobable or when there is no information on the distribution between these limits.

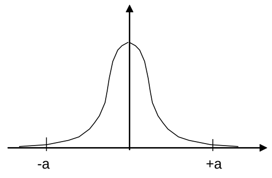


- In this case the relation between **standard uncertainty u** and the limits of variation (maximum deviation ± a) is:

$$\sigma^2 = \int_{-\infty}^{\infty} p(x)(x - x_m)^2 dx = \int_{-a}^{a} \frac{1}{2a} x^2 dx = \frac{1}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{a^2}{3}$$

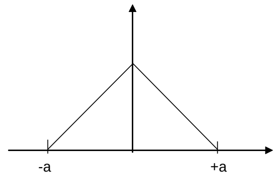
$$u = \frac{a}{\sqrt{3}} \cong 0.6a$$

Other Distributions



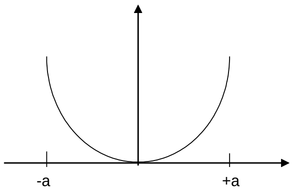
Normal distribution

$$u = \frac{a}{2} = 0.5a$$



Triangular distribution

$$u = \frac{a}{\sqrt{6}} \cong 0.4a$$



U shaped distribution

$$u = \frac{a}{\sqrt{2}} \cong 0.7a$$

Uncertainty for Direct Measurements

- Direct measurements are distinguished between **single measurement and repeated measurements**.
- Single measurement is usually adopted when using methods and tools that are not too "sensitive", so that one always expects to obtain the same result
- Repeated measurements are adopted with instruments and methods so "sensitive" as to highlight the variations induced on the measurement by the influence quantities.

Single Measurements

- In single measurements, uncertainty is obtained, after correcting any systematic errors, **by combining the type B uncertainties due to the instrumentation and other causes**.

- For **the standard uncertainty** we have:

$$u_y = \sqrt{\sum_{i=1}^{N} u_i^2}$$

Repeated Measurement

- In repeated measurement the best estimate is given by the average of the various measurements and the total uncertainty must be calculated by combining random and systematic uncertainties. :

- standard uncertainty

$$u_{tot} = \sqrt{u_A^2 + u_B^2}$$

Uncertainty for Indirect Measurements (uncertainty propagation)

- In some cases, the y measurand is not predictable by a direct measure but is a function of other N quantities x_i related to it through a functional relationship
- In the case of three input sizes we can write:

$$y = f(a,b,c)$$

- For the input quantities, the value will be known with an associated uncertainty; then we have:

$$a_m = a_0(u_a) \qquad b_m = b_0(u_b) \qquad c_m = c_0(u_c)$$

$$y_0 = f(a_0, b_0, c_0)$$ value of the measurement

Probabilistic Evaluation of Uncertainty for Indirect Measurements (uncertainty propagation)

If a probabilistic evaluation is required, in the absence of correlation between the quantities, the variance can be calculated as:

$$\sigma_y^2 = \left(\frac{\partial f}{\partial a}\Big|_{a_0,b_0,c_0}\right)^2 \sigma_a^2 + \left(\frac{\partial f}{\partial b}\Big|_{a_0,b_0,c_0}\right)^2 \sigma_b^2 + \left(\frac{\partial f}{\partial c}\Big|_{a_0,b_0,c_0}\right)^2 \sigma_c^2$$

This is the model that must be used in the estimate uncertainties in issuing official certificates;
it is also the model suggested by the Guide of measurement uncertainty (CEI UNI).
Moving from variances to uncertainties we have:

$$u_y^2 = \left(\frac{\partial f}{\partial a}\Big|_{a_0,b_0,c_0}\right)^2 u_a^2 + \left(\frac{\partial f}{\partial b}\Big|_{a_0,b_0,c_0}\right)^2 u_b^2 + \left(\frac{\partial f}{\partial c}\Big|_{a_0,b_0,c_0}\right)^2 u_c^2$$

Expanded Uncertainty

- Since more quantities are involved, the distribution tends to the Gaussian one and σ_y assumes the meaning of u_y-like uncertainty with a confidence of 68.4%.

- If you want to have higher probabilities you need to multiply u_y by a coverage factor:

$$U = K \cdot U_y$$