REVIEW OF SCATTERING MATRIX & COMPONENTS

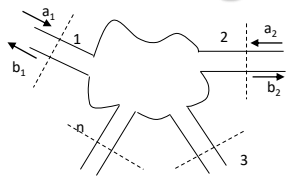
1) **Scattering Matrix**
   a. Definition
   b. Properties

2) **Components**
   a. Two ports (attenuators, filters, amplifiers)
   b. Three ports (dividers, circulators)
   c. Four ports (directional couplers)

Scattering-Matrix Definition and Properties

Voltage waves

N-Port Network



a_i = incident waves
b_i = reflected waves

$$V_i = a_i + b_i \qquad I_i = (a_i - b_i)/Z_{0i}$$

Voltage waves

$$\mathbf{a}_i = \frac{\mathbf{V}_i + \mathbf{I}_i Z_{0i}}{2} \qquad \mathbf{b}_i = \frac{\mathbf{V}_i - \mathbf{I}_i Z_{0i}}{2}$$

Z_{0i} = Transmission Line Characteristic Impedance
Tipically Z_{0i} = 50 Ω

Why 50 Ohms?

Why was 50 ohms selected as the reference impedance of radar and radio equipment? There are many misconceptions surrounding this bit of history. "The first coaxial dimensions just happened to define the 50 ohm reference impedance." "Fifty ohms matched up well with the antennas in use 60 years ago." In actuality, the 50 ohm reference impedance was selected from a trade-off between the lowest loss and maximum power-handling dimension for an air line coaxial cable. The optimum ratio of the outer conductor to inner conductor, for minimum attenuation in a coaxial structure with air as the dielectric, is 3.6. This corresponds to an impedance, Zo, of 77 ohms.[5] Although this yields the best performance from a loss standpoint, it does not provide the maximum peak power handling before dielectric breakdown occurs. The best power performance is achieved when the ratio of the outer conductor to inner conductor is 1.65. This corresponds to a Zo of 30 ohms.[5] The geometric mean of 77 ohms and 30 ohms is approximately 50 ohms [Eq. (6.41)]; thus, the 50 ohm standard is a compromise between best attenuation performance and maximum peak power handling in the coaxial cable.

$$50 \approx \sqrt{30 * 77} \qquad (6.41)$$

Scattering matrix for a N-Port Network

For linear networks holds the **superposition principle**

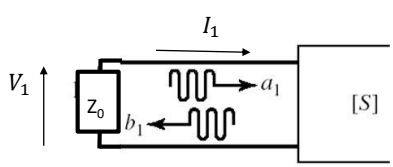
$$\mathbf{b}_1 = S_{11}\mathbf{a}_1 + S_{12}\mathbf{a}_2 + \ldots + S_{1n}\mathbf{a}_n$$
$$\mathbf{b}_2 = S_{21}\mathbf{a}_1 + S_{22}\mathbf{a}_2 + \ldots + S_{2n}\mathbf{a}_n$$
$$\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathbf{b}_n = S_{n1}\mathbf{a}_1 + S_{n2}\mathbf{a}_2 + \ldots + S_{nn}\mathbf{a}_n$$

$$\begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_n \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \ldots\ldots & S_{1n} \\ S_{21} & S_{22} & \ldots\ldots & S_{2n} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ S_{n1} & S_{n2} & \ldots\ldots & S_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \\ a_n \end{bmatrix}$$

[**b**] = [S] [**a**]

$$S_{ij} = \left(\frac{\mathbf{b}_i}{\mathbf{a}_j}\right)_{\mathbf{a}_k = 0 \text{ con } k \neq j}$$

Matched Load



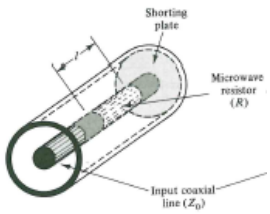
$$V_1 = -Z_0 I_1$$

$$a_1 = \frac{V_1 + Z_0 I_1}{2} = 0$$

When a network is closed on an matched load (50 Ω) the wave reflected by the load and incident on the port (a_1) is equal to zero

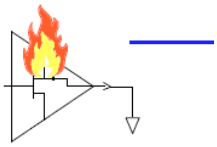
Why scattering parameters ?

- Scattering parameters can be measured at microwave frequencies as they require the network to be closed on 50 Ohm and this kind of loads are "relatively" easy to realize



Why scattering parameters ?

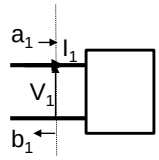
- At microwave frequencies, voltages and currents are hard to measure and the open and short circuits necessary to measure impedance and admittance parameters can cause the destruction of active devices



- The scattering parameters are defined for all two-port networks while the admittance and impedance parameters are not defined in some cases. For example, for a series impedance the impedance parameters are not defined because the currents at the two ports are not independent

Scattering matrix for 1 Port Network

E = Incoming
I = Incident
R = Reflected



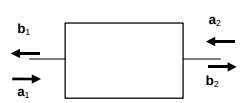
$$P_1 = P_E = \frac{1}{2}\mathrm{Re}\left(V_1 I_1^*\right) = \frac{1}{2}\mathrm{Re}\left\{\left(a_1 + b_1\right)\left(\frac{a_1 - b_1}{Z_0}\right)^*\right\} = \frac{1}{2Z_0}|a_1|^2 - \frac{1}{2Z_0}|b_1|^2$$

$$P_1 = P_E = P_I - P_R = P_I\left(1 - \frac{P_R}{P_I}\right) = P_I\left(1 - \frac{|b_1|^2}{|a_1|^2}\right)$$

$$|S|^2 = |\Gamma_I|^2 = \frac{|b_1|^2}{|a_1|^2} = \frac{P_R}{P_I}$$

the square modules of the scattering Parameters are ratios between powers

Scattering matrix for 2 Port Network



$$P_2 = \frac{1}{2}\mathrm{Re}\left(V_2(-I_2)^*\right) = \frac{1}{2}\mathrm{Re}\left\{\left(a_2 + b_2\right)\left(\frac{b_2 - a_2}{Z_0}\right)^*\right\} = -\frac{1}{2Z_0}|a_2|^2 + \frac{1}{2Z_0}|b_2|^2$$

if $a_2 = 0 \longrightarrow P_2 = P_O = \frac{1}{2Z_0}|b_2|^2 = \frac{1}{2Z_0}|S_{21}|^2|a_1|^2 = P_I|S_{21}|^2 \longrightarrow \boxed{|S_{21}|^2 = \frac{P_O}{P_I}}$

if $a_2 = 0 \longrightarrow |\Gamma_I| = |S_{11}| \longrightarrow \boxed{|S_{11}|^2 = \frac{P_R}{P_I}}$

the square modules of the scattering Parameters are ratios between powers

Reciprocal two port network

(Absence of saturated ferrites or controlled generators in the component under test)

$$[Z]=[Z]^T \rightarrow [S]=[S]^T$$

if [S] is defined starting from voltage waves, this is true if all the Z_{0i} are equal

$$s_{21} = s_{12}$$

Symmetrical two port network

A two port network is symmetrical if its input impedance is equal to its output impedance

Symmetrical networks are also physically symmetrical

$$S_{11} = S_{22}$$

Lossless two port network

$$[S]^{T*}[S]=[1] \qquad [S]=\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \qquad [S^T]=\begin{bmatrix} s_{11} & s_{21} \\ s_{12} & s_{22} \end{bmatrix}$$

$$|s_{11}|^2+|s_{12}|^2=1 \qquad |s_{21}|^2+|s_{22}|^2=1$$

$$|s_{11}|\,|s_{21}|e^{j(\vartheta_{11}-\vartheta_{21})}+|s_{12}|\,|s_{22}|e^{j(\vartheta_{12}-\vartheta_{22})}=0 \qquad \begin{cases} |s_{11}|\,|s_{21}|=|s_{12}|\,|s_{22}| \\ \vartheta_{11}-\vartheta_{21}=\pi+\vartheta_{12}-\vartheta_{22} \end{cases}$$

$$|s_{21}| = |s_{12}| \qquad |s_{11}| = |s_{22}| \qquad |s_{12}|=\sqrt{1-|s_{11}|^2}$$

Raised components

$$\sum_{j=1}^{n} S_{ij} = 1 \qquad i=1,n$$

$$\sum_{i=1}^{n} S_{ij} = 1 \qquad j=1,n$$

Parameters of two-port passive structures

Reflection attenuation

$$A_{RdB} = 10\log_{10}\frac{P_I}{P_E} = 10\log_{10}\frac{1}{1-|S_{11}|^2}$$

Dissipation attenuation

$$A_{DdB} = 10\log_{10}\frac{P_E}{P_O} = 10\log_{10}\frac{1-|S_{11}|^2}{|S_{21}|^2}$$

Total Attenuation or Insertion Loss (positive number)

$$A_{dB} = 10\log_{10}\frac{P_I}{P_O} = 10\log_{10}\frac{1}{|S_{21}|^2} = A_{RdB} + A_{DdB}$$

Return loss (positive number)

$$L_{RdB} = 10\log_{10}\frac{1}{|S_{11}|^2}$$

Measured Parameters

S_{11} scattering parameter

$$S_{11dB} = 10\log_{10}|S_{11}|^2 = 20\log_{10}|S_{11}|$$

Return loss

$$L_{RdB} = 10\log_{10}\frac{1}{|S_{11}|^2} = -S_{11dB}$$

S_{21} scattering parameter

$$S_{21dB} = 10\log_{10}|S_{21}|^2 = 20\log_{10}|S_{21}|$$

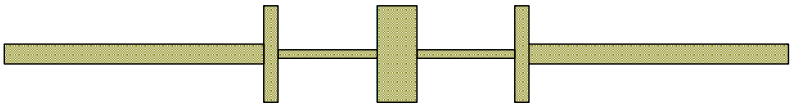
Total attenuation

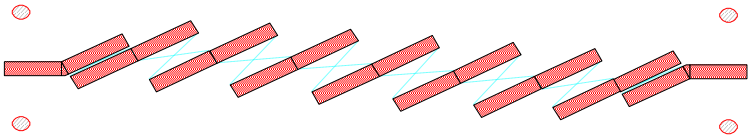
$$A_{dB} = 10\log_{10}\frac{1}{|S_{21}|^2} = -S_{21dB}$$

Components

Microstrip Filters

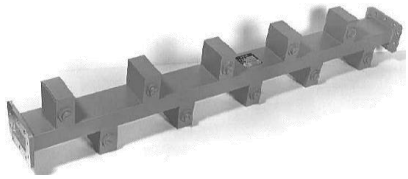
Low pass 6 GHz





Band pass 17-19 GHz

Waveguide and Coaxial Cable Filters

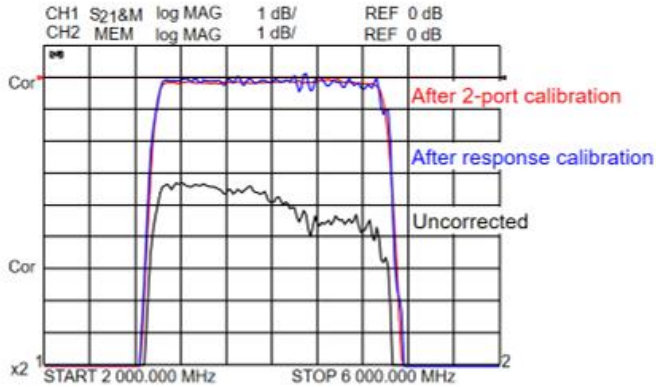


Pass band 19.73-21.1 GHz

Ceramic
Pass band 2.2-2.3 GHz

Response versus Two-Port Calibration

Measuring filter insertion loss



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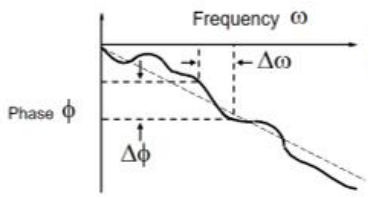
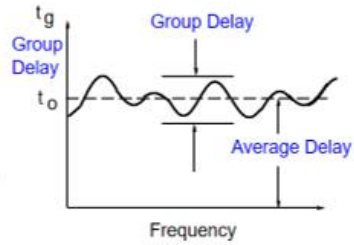
What is group delay?





$$\text{Group Delay } (t_g) = \frac{-d\phi}{d\omega} = \frac{-1}{360^\circ} * \frac{d\phi}{d_f}$$

ϕ in radians

ω in radians/sec

ϕ in degrees

f in Hz ($\omega = 2\pi f$)

Deviation from constant group delay indicates distortion

Average delay indicates transit time

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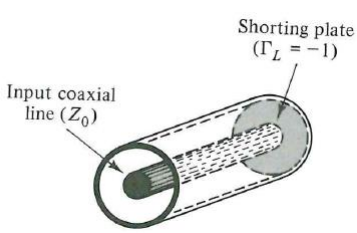
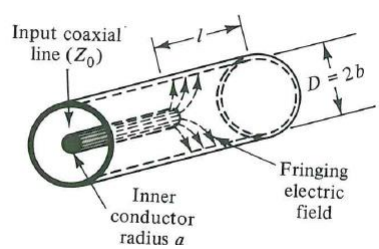
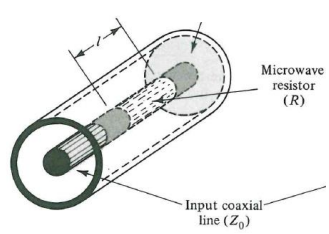
Loads



Short



Open



50 Ohm

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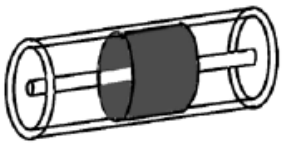
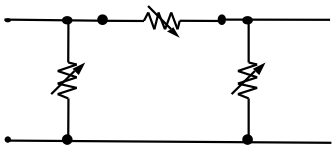
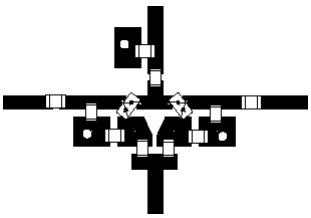
Coaxial cable Attenuators



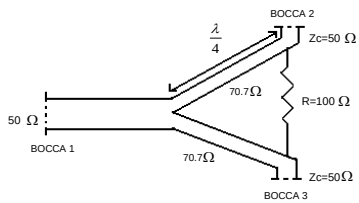
Pin diode attenuators





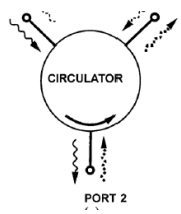
Three port passive structures (dividers and circulators)

Divider Wilkinson



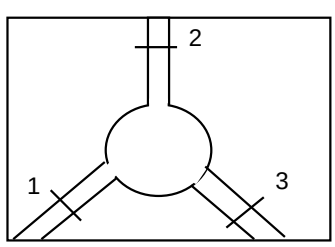
$$[S]=\begin{bmatrix} 0 & -j\frac{1}{\sqrt{2}} & -j\frac{1}{\sqrt{2}} \\ -j\frac{1}{\sqrt{2}} & 0 & 0 \\ -j\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

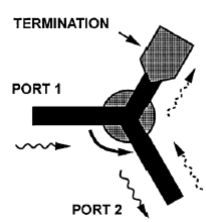
Circulator

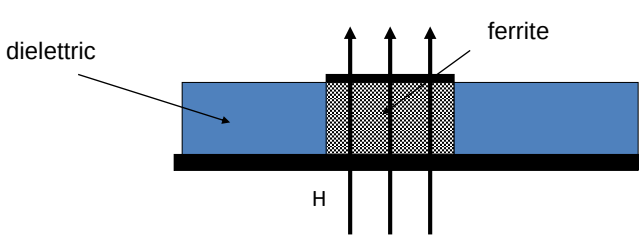


$$[S]=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

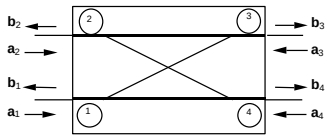
Ferrite circulator







AD Parameters



Coupling	$C_{dB} = 10\log\frac{P_I}{P_{31}}$
Isolation	$I_{dB} = 10\log\frac{P_I}{P_{21}}$
Directivity	$D_{dB} = 10\log\frac{P_{31}}{P_{21}}$

With ports closed on matched loads: $a_2 = 0$, $a_3 = 0$, $a_4 = 0$
and port 1 powered by a matched generator

.

Positive number

$$C_{dB} = -10\log_{10}|S_{31}|^2$$

Positive number

$$I_{dB} = -10\log_{10}|S_{21}|^2$$

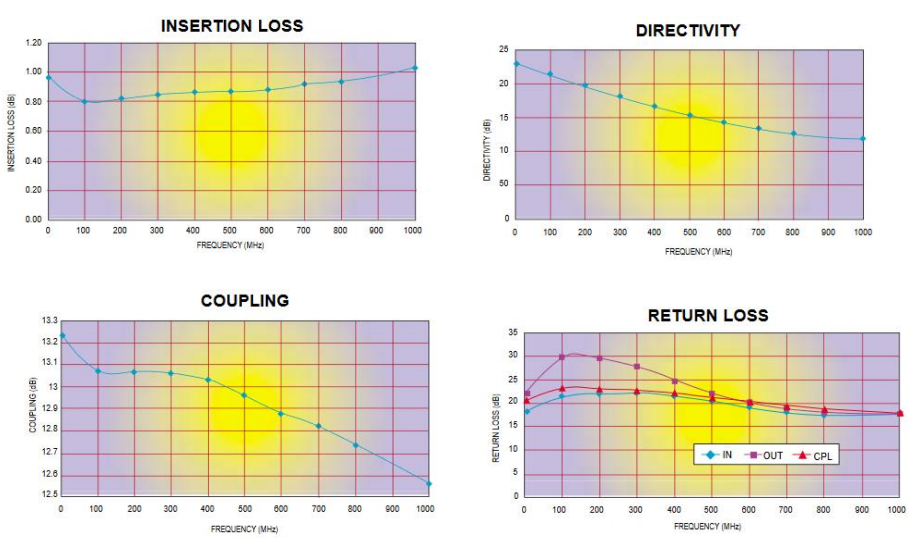
Positive number

$$D_{dB} = 10\log_{10}\frac{|S_{31}|^2}{|S_{21}|^2}$$

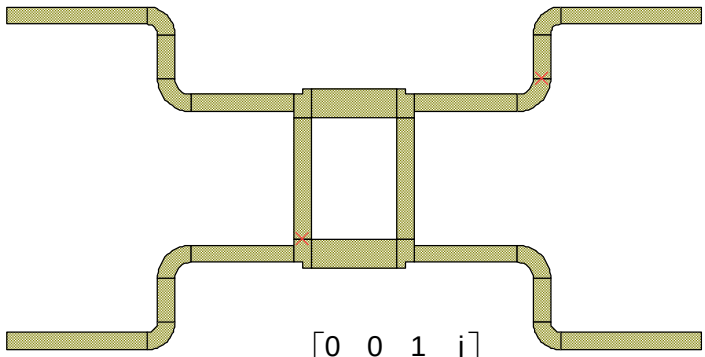
$D = P_{31}/P_{21} = (P_{31}/P_I)\,(P_I/P_{21}) =$

$$D_{dB} = I_{dB} - C_{dB}$$

AD Parameters

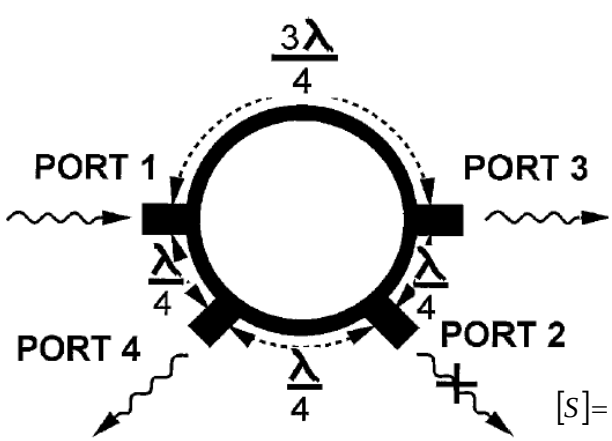


Microstrip 90° DC (Branch Coupler)



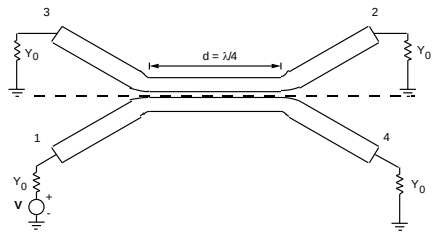
$$[S]=\frac{-1}{\sqrt{2}}\begin{bmatrix}0 & 0 & 1 & j\\ 0 & 0 & j & 1\\ 1 & j & 0 & 0\\ j & 1 & 0 & 0\end{bmatrix}$$

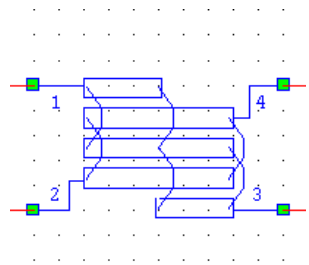
Microstrip 180° DC (rat race)



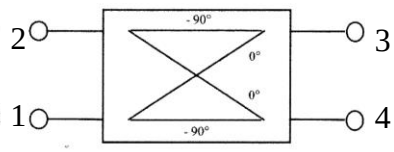
$$[S]=\frac{-j}{\sqrt{2}}\begin{bmatrix}0 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\\ -1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\end{bmatrix}$$

Microstrip Distributed DC





Lange DC 90° hybrid



hybrid => power divider

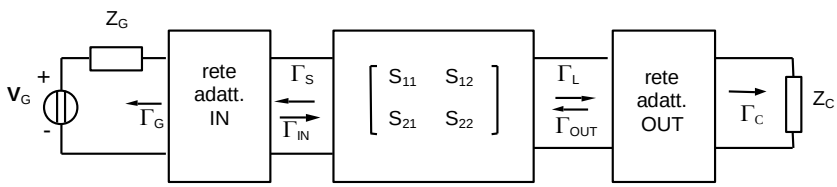
$$|s_{13}| + |s_{14}| = \frac{1}{\sqrt{2}}$$

90°=>

phase(S_{13})-phase (S_{14}) = -90

Coaxial distributed DC

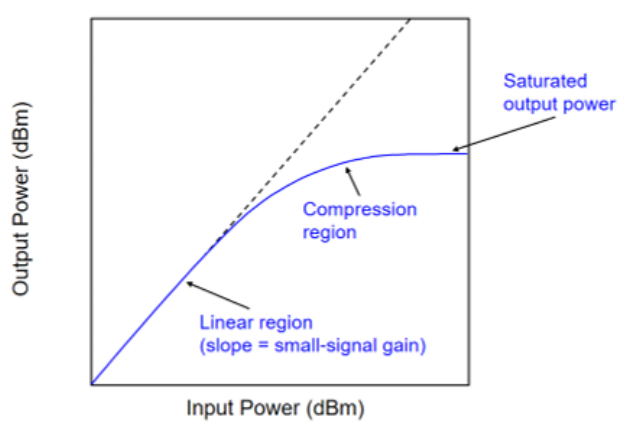
Amplifiers



$$G_T = |S_{21}|^2$$

$$R_L = |S_{11}|^2$$

Power Sweep - Compression



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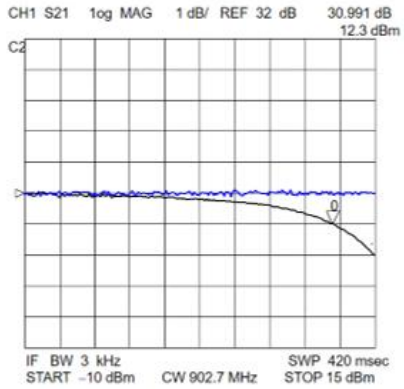
Power Sweep -Gain Compression



1 dB compression: input power resulting in 1 dB ***drop*** in gain

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