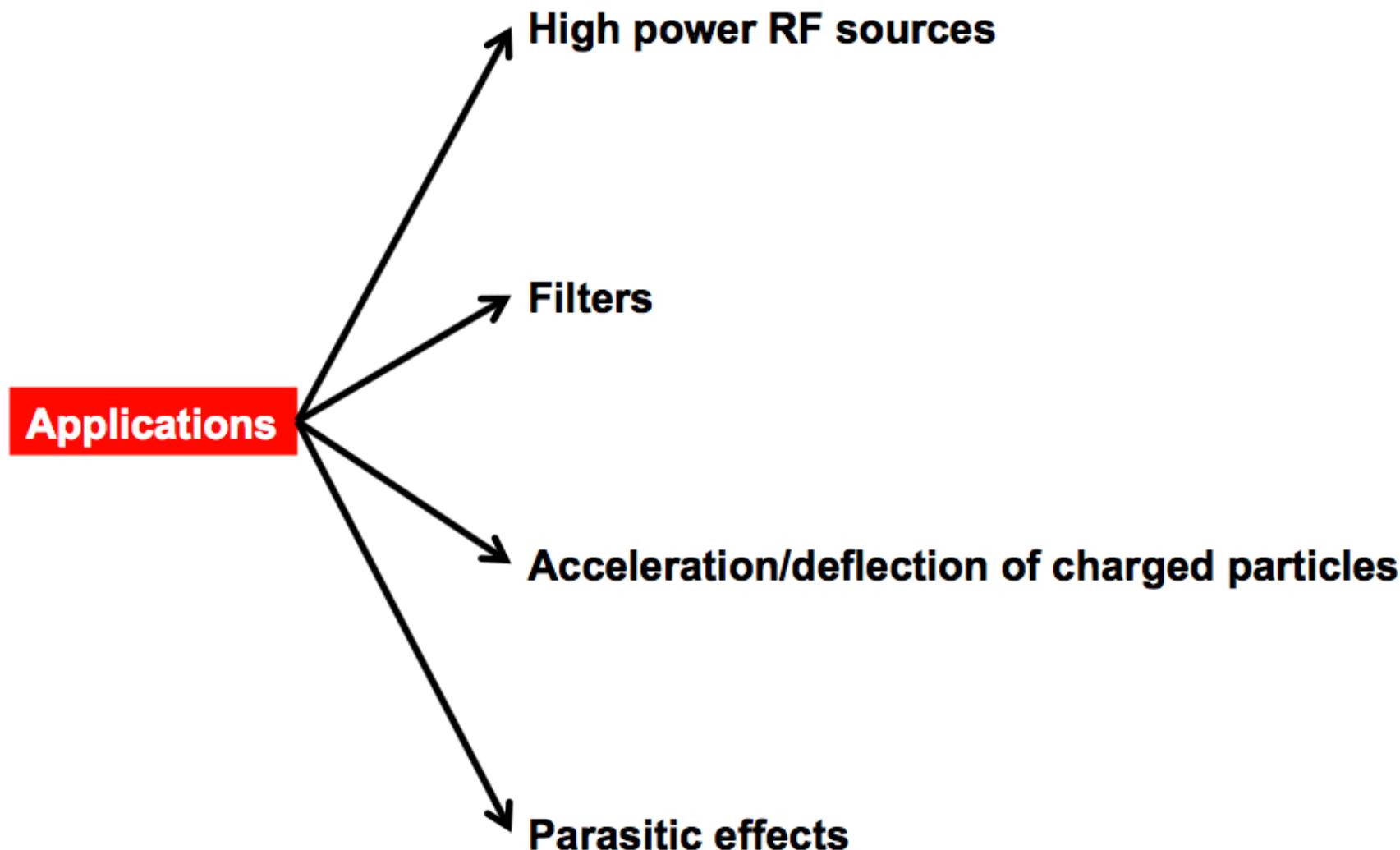


# Resonant cavities



# Resonant cavities

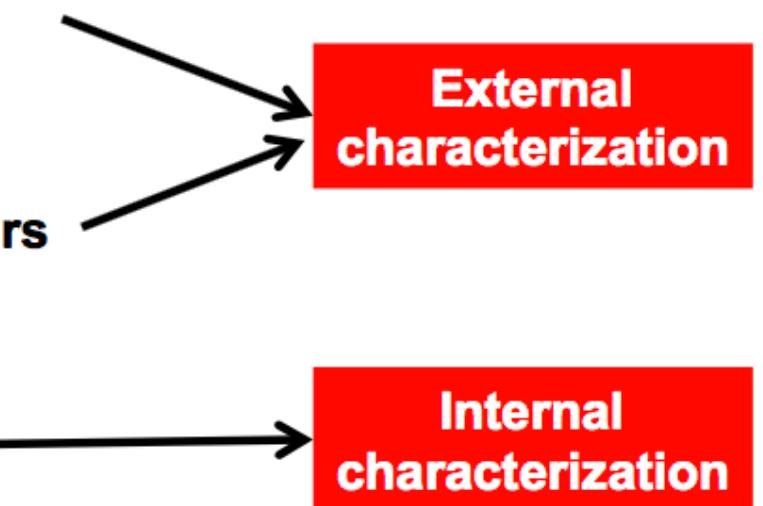
Modes of pillbox cavities

Equivalent circuit for a mode

Reflection measurement of circuit parameters

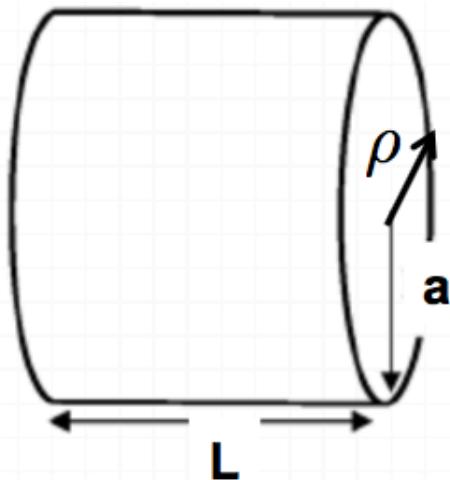
Transmission measurement of circuit parameters

Field measurement inside the cavity



Examples of multi-cell cavities

# Pillbox cavity



Fundamental mode

$$L/a < 2.03$$

**TM010**

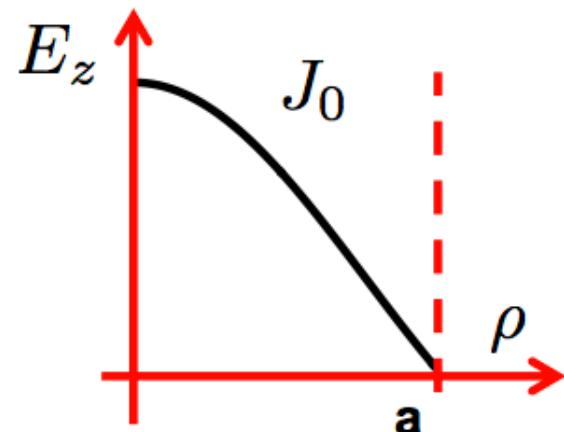
$$L/a > 2.03$$

**TE111**

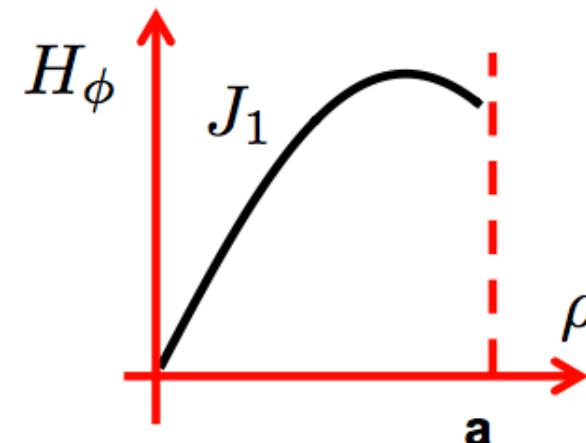
$$E_z = E_0 J_0 \left( \frac{\chi_{01}}{a} \rho \right)$$

**TM010**

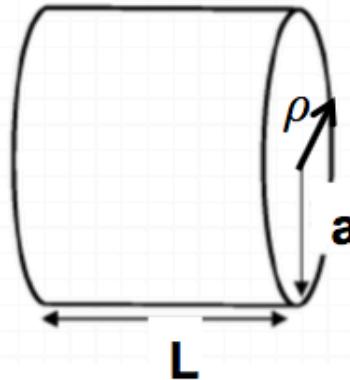
$$Z_0 H_\phi = j E_0 J_1 \left( \frac{\chi_{01}}{a} \rho \right)$$



$$\chi_{n,m} : J_n(\chi_{n,m}) = 0$$



# Pillbox cavity



Average stored energy

$$W = 2 \left( \frac{1}{4} \epsilon_0 \int_{\tau} |E|^2 d\tau \right)$$

Average dissipated power

$$P_d = \frac{R_s}{2} \oint_S |H|^2 dS$$

$$Q = \frac{\omega_0 W}{P_d} \quad R_s = \sqrt{\frac{\omega_0 \mu}{2\sigma}} = \frac{1}{\sigma \delta_{skin}}$$

TM010



$$Q = \frac{\omega_0 W}{P_d} = \frac{1.2025 Z_{vacuum}}{R_s (1 + a/L)}$$

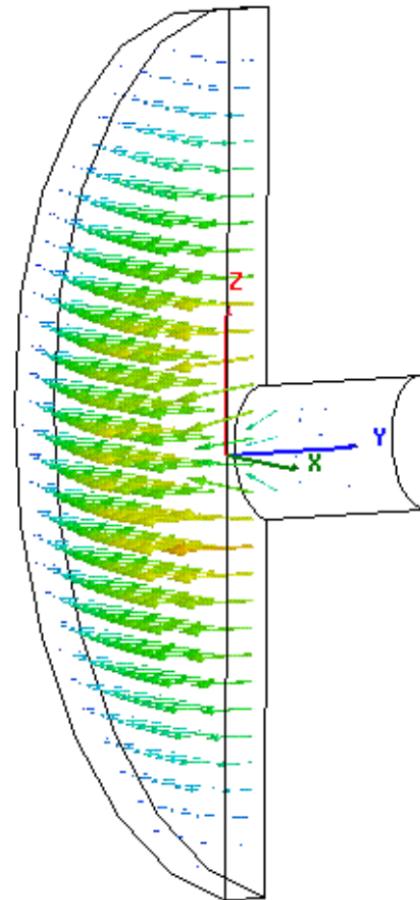
$a = 6 \text{ cm}$

$L = 4.3 \text{ cm}$

Compute the resonant frequency of  
the modes up to 6GHz

# Pillbox: TM010

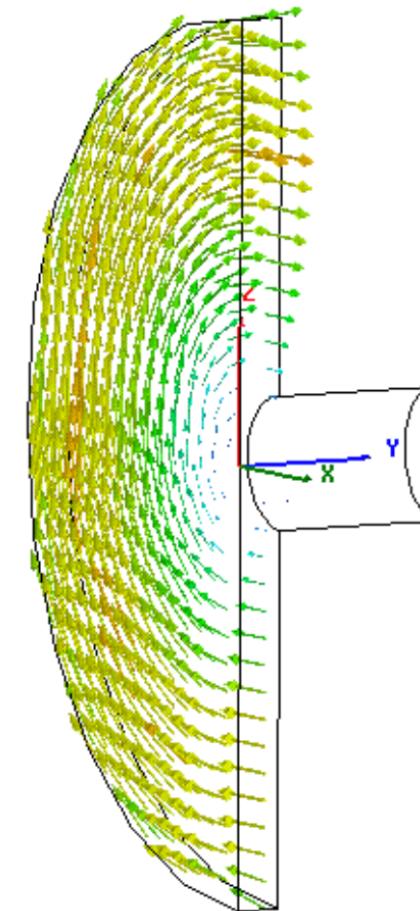
a cavity radius



electric field

L cavity length

TM<sub>010</sub>-mode  
(only 1/4 shown)



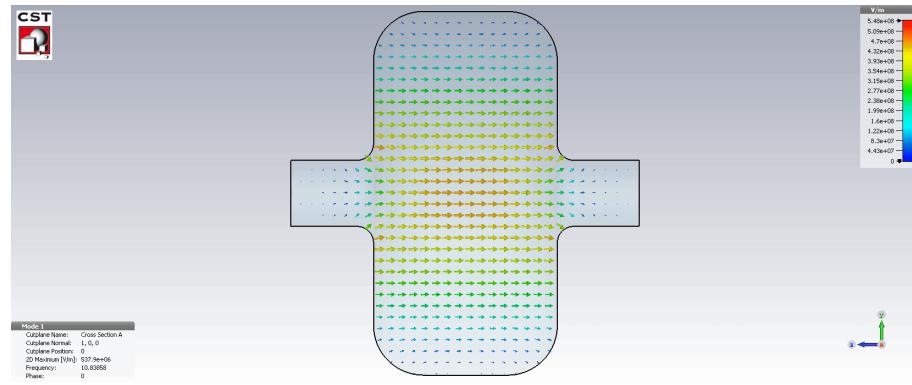
magnetic field

$$E_z = E_0 J_0 \left( \frac{\chi_{01}}{a} \rho \right)$$

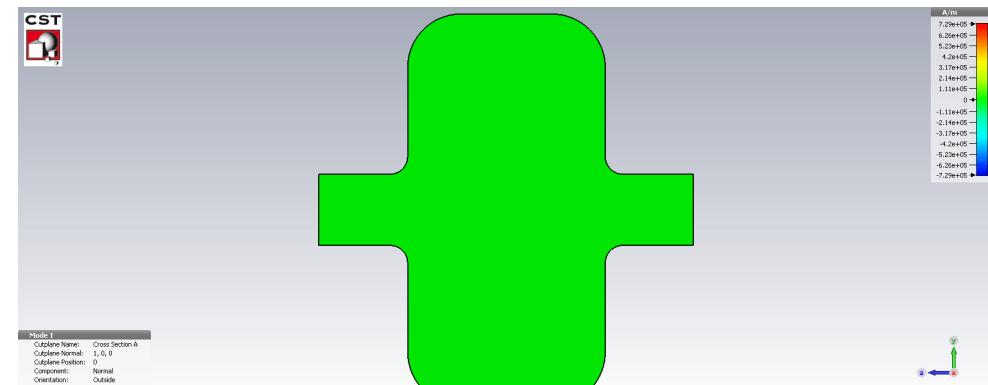
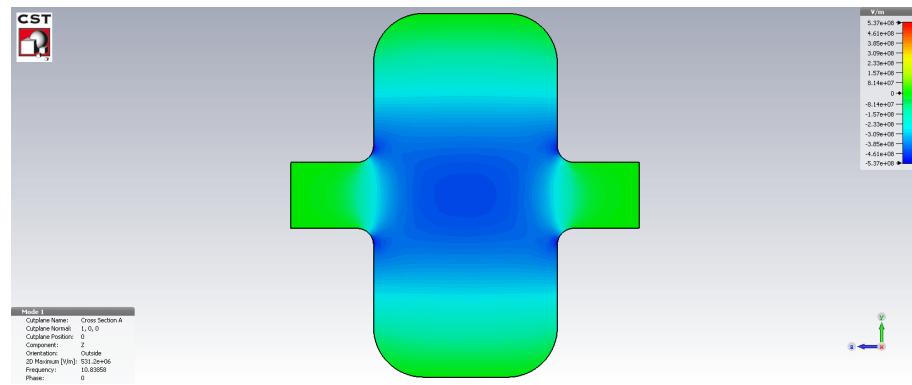
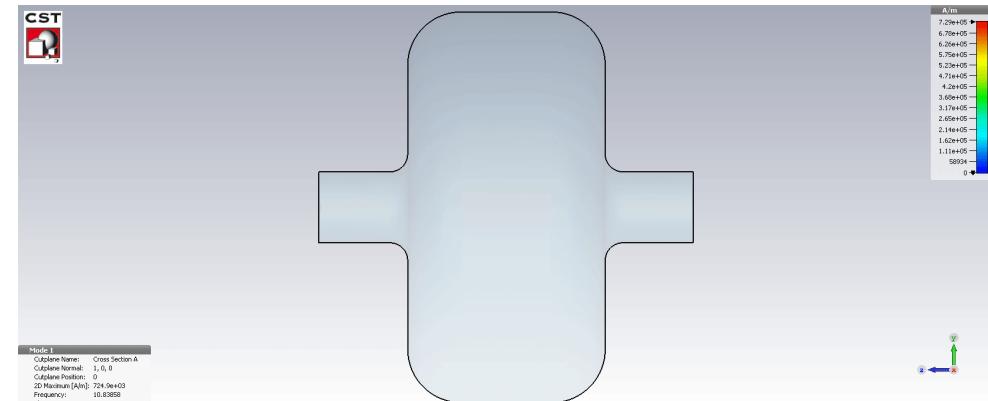
$$Z_0 H_\phi = j E_0 J_1 \left( \frac{\chi_{01}}{a} \rho \right)$$

# Pillbox: TM010

a cavity radius



L cavity length

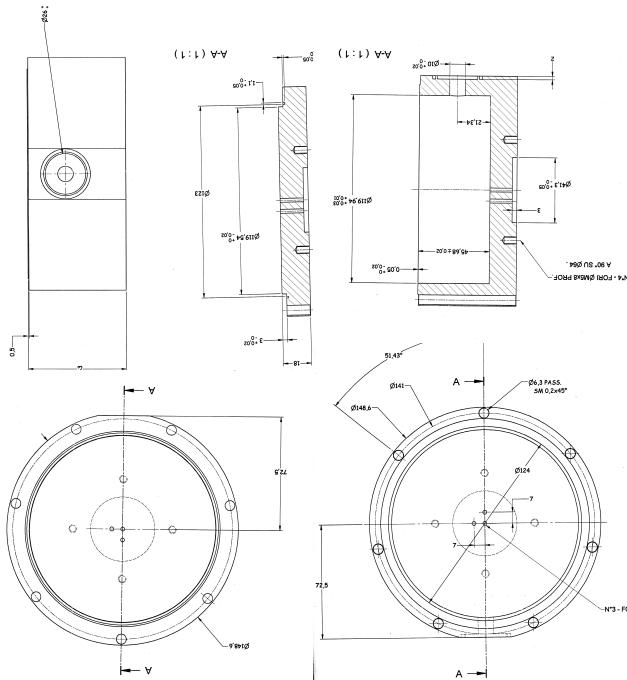
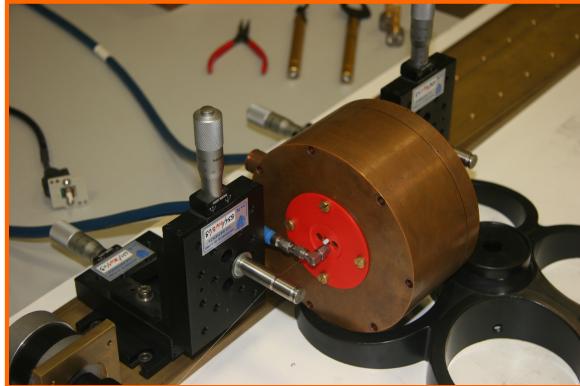


$$E_z = E_0 J_0 \left( \frac{\chi_{01}}{a} \rho \right)$$

$$Z_0 H_\phi = j E_0 J_1 \left( \frac{\chi_{01}}{a} \rho \right)$$

# RF properties of pillbox cavity (HFSS simulation)

## Numerical results

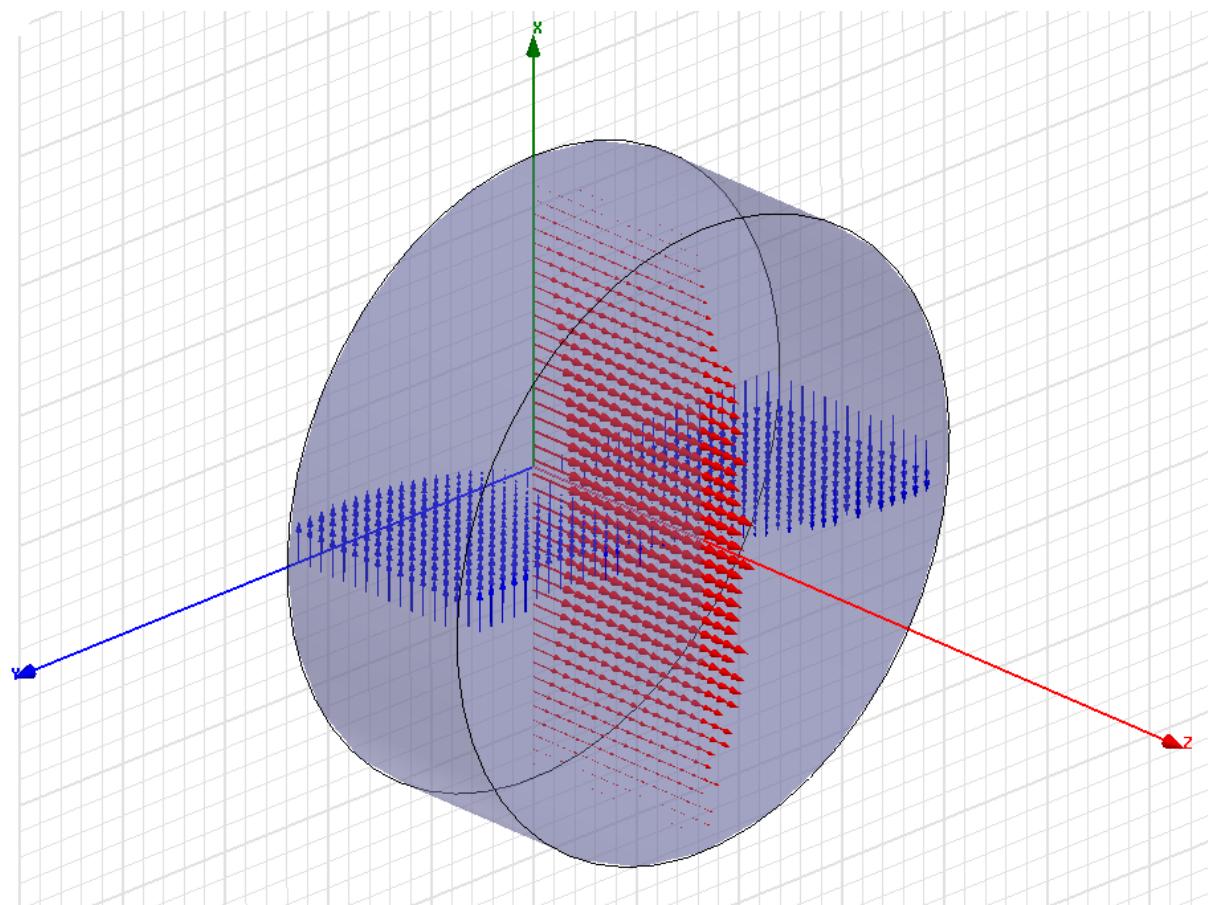
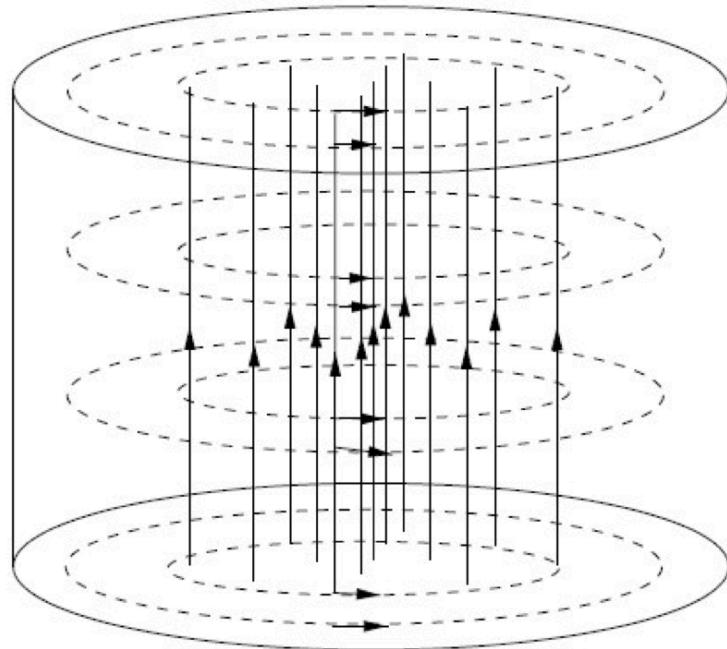
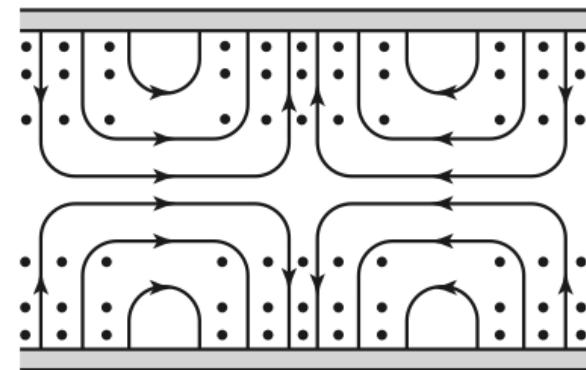
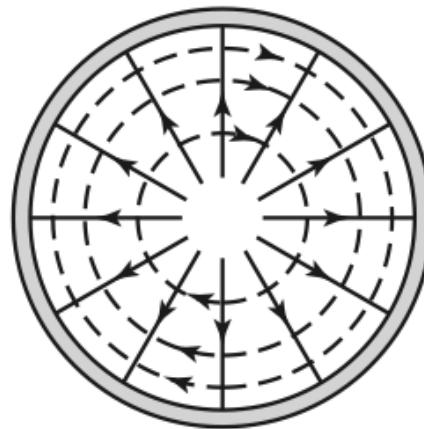
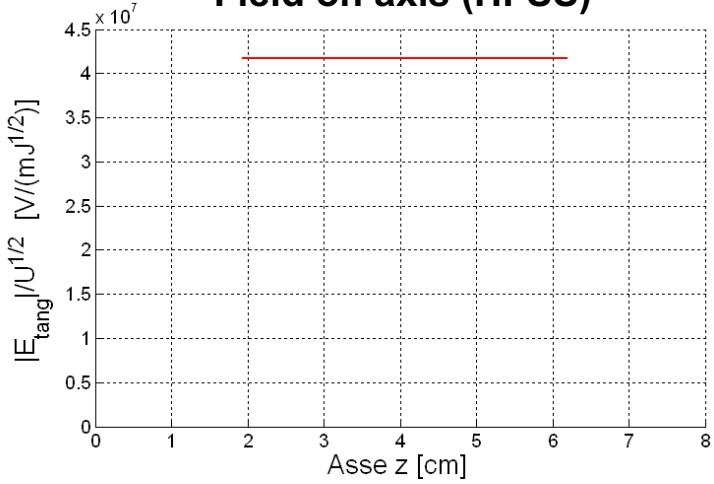


| Cavity mode | F res [GHz] | E/(U <sup>1/2</sup> ) [V/(mJ <sup>1/2</sup> )] | H/(U <sup>1/2</sup> ) [A/(mJ <sup>1/2</sup> )] |
|-------------|-------------|--|--|
| TM010       | 1.9133      | Tang:4.1681e+07                                | /  |
| TM110 (0)   | 3.0486      | /  | Norm:1.0085e+05                                |
| TM110 (90)  | 3.0486      | /  | Norm:1.0083e+05                                |
| TE111 (0)   | 3.8036      | Norm:4.4153e+07                                | Norm:1.0836e+05                                |
| TE111 (90)  | 3.8037      | Norm:4.4191e+07                                | Norm:1.0836e+05                                |
| TM011       | 3.9962      | Tang:2.8208e+07                                | /  |
| *TM210 (0)  | 4.086       | /  | /  |
| *TM210 (90) | 4.086       | /  | /  |
| *TE211 (0)  | 4.2672      | /  | /  |
| *TE211 (90) | 4.2676      | /  | /  |

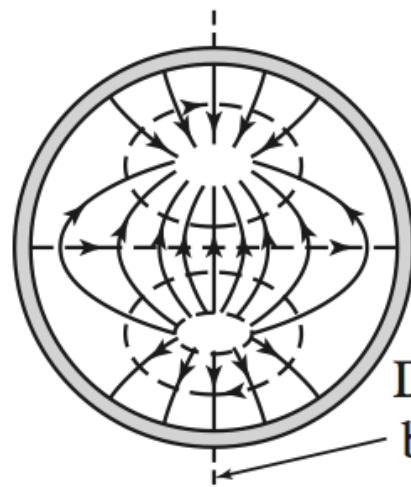
\* Mode with zero electric field on axis.

## TM010 mode

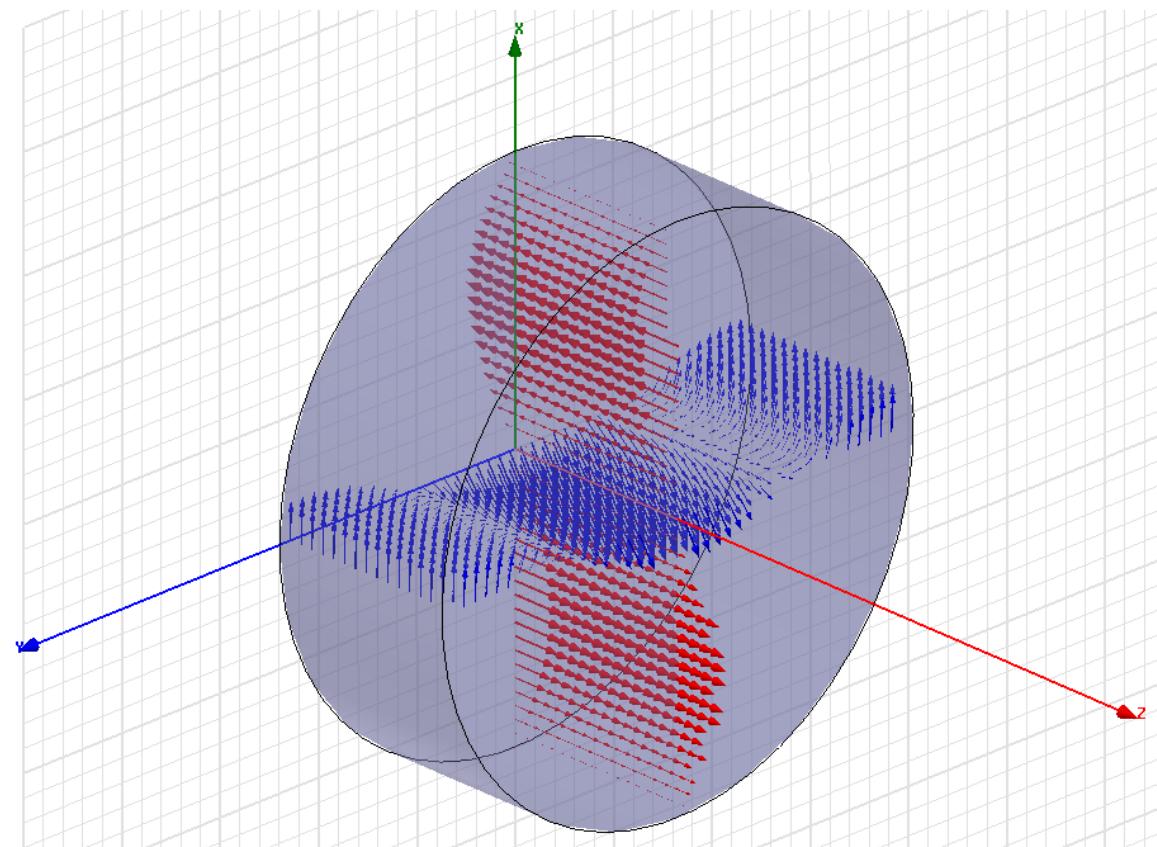
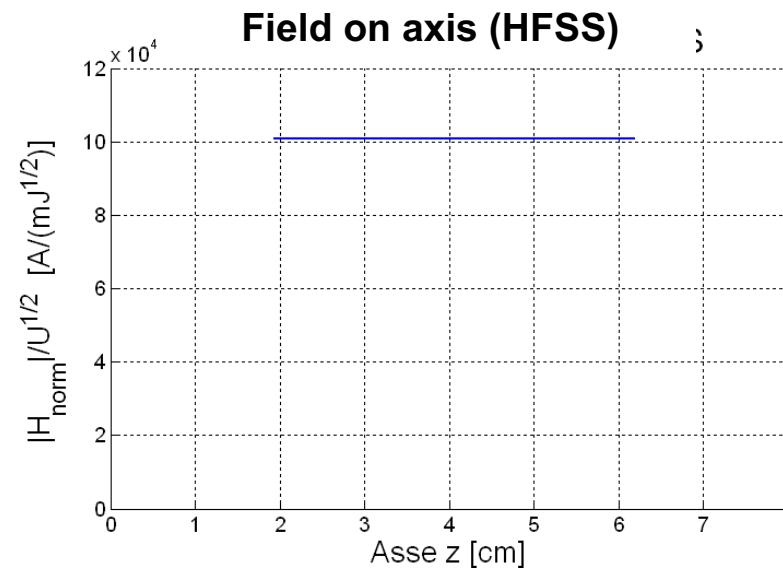
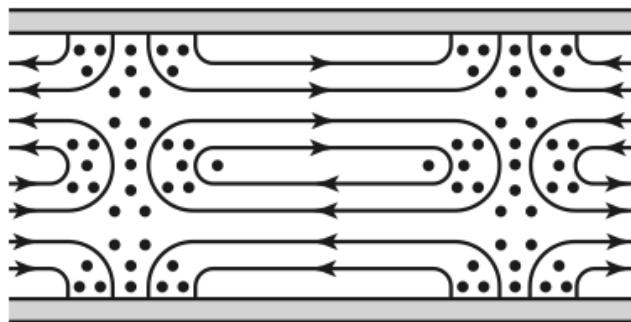
**Field on axis (HFSS)**



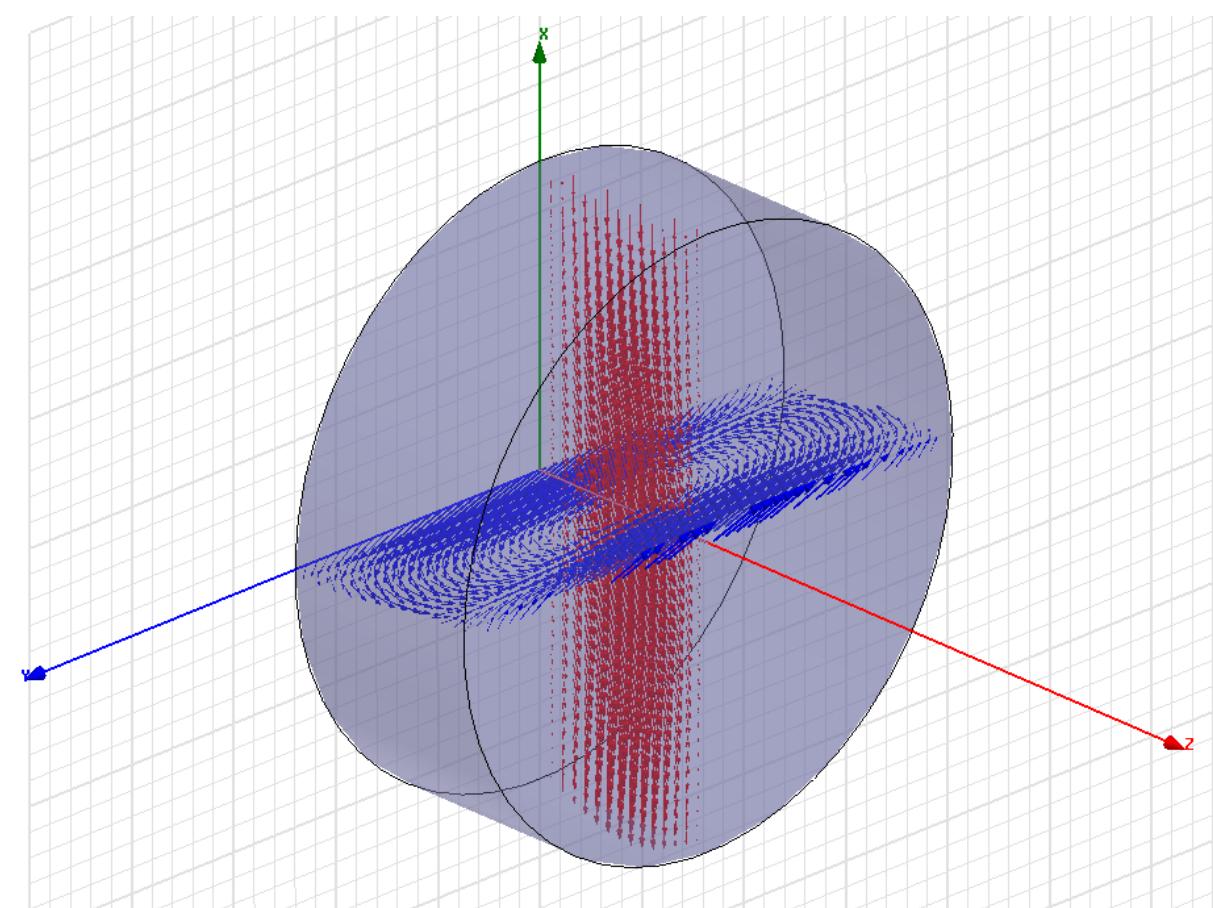
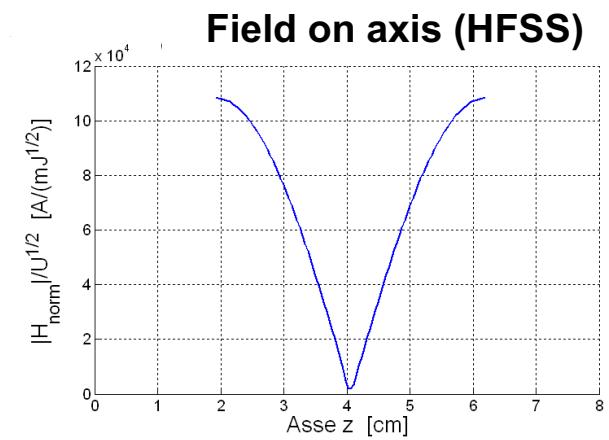
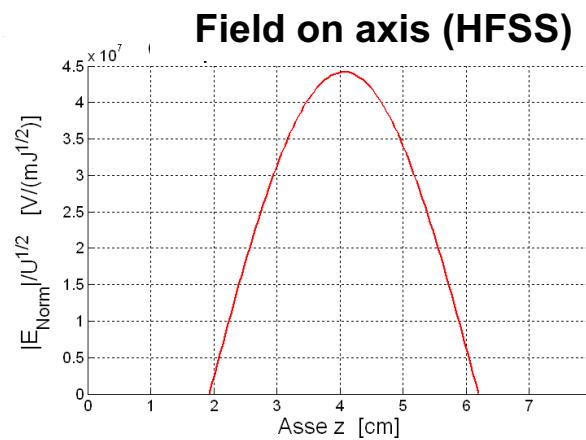
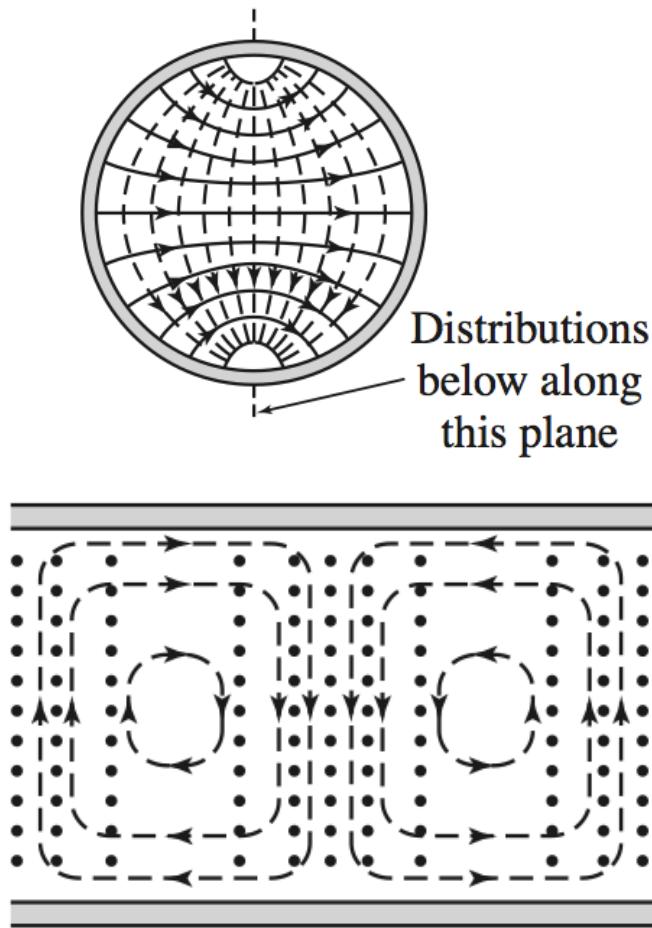
## TM110 mode



Distributions  
below along  
this plane



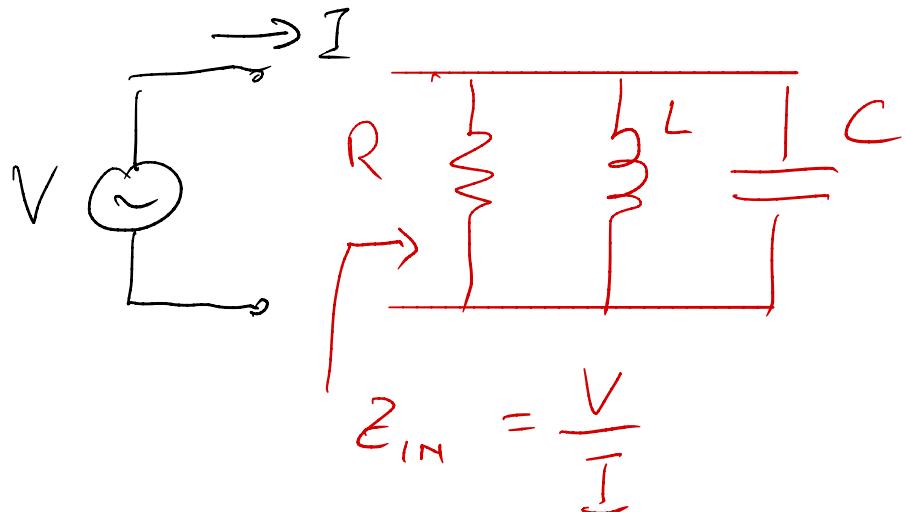
## TE111 mode



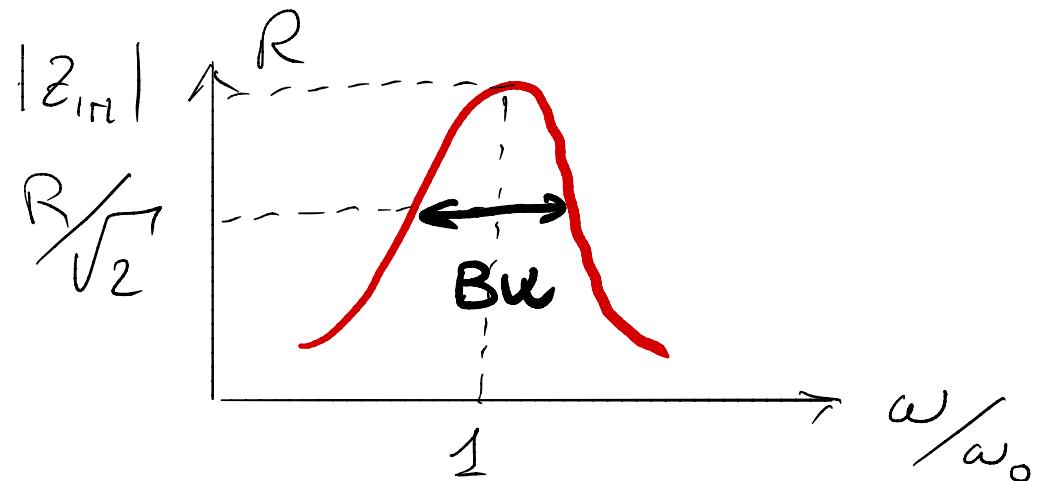
# EQUIVALENT CIRCUIT OF CAVITY MODES

SERIES

CIRCUIT OR



PARALLEL CIRCUIT



$$Z_{IN} = \left( \frac{1}{R} + \frac{1}{j\omega L} + j\omega C \right)^{-1} = \frac{R}{1 + j Q_0 \delta}$$

$\omega \approx \omega_0$

$$\delta = \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \stackrel{\downarrow}{\approx} 2 \frac{(\omega - \omega_0)}{\omega_0}$$

$$Z_{IN} \approx \frac{R}{1 + 2 j Q_0 (\omega - \omega_0)/\omega_0}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

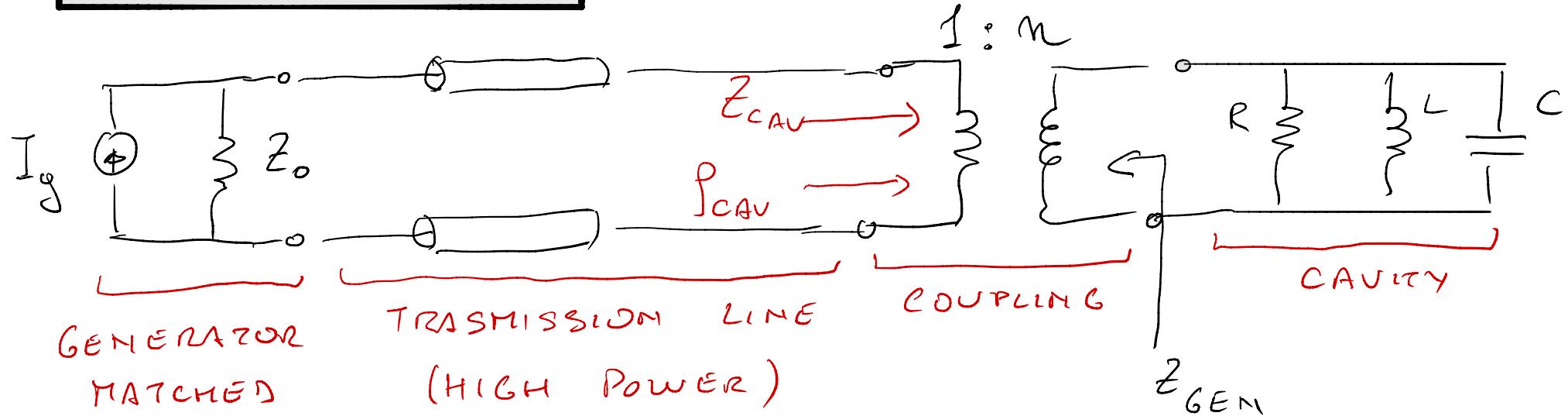
$$\frac{\Delta \omega_{3dB}}{\omega_0} = \frac{Bw}{\omega_0} = \frac{1}{Q_0}$$

$$Q_o = \omega_o \frac{W_E + W_H}{P_{Cav}} = \frac{R}{\omega_o L} = R C \omega_o$$

*C → W\_E + W\_H ← L*

*R → P\_{Cav}*

## CAVITY COUPLING



$$Z_{cav} = \frac{R/m^2}{1 + j Q_o \delta}$$

$$Q_o = \omega_o \frac{W}{P_{cav}}$$

W STORED ENERGY (AVERAGE ON A PERIOD OF ...)

P<sub>cav</sub> POWER LOSS IN CAVITY WALLS (AVERAGE ON ...)

P<sub>ext</sub> POWER LOSS ON  $Z_0$  IRRADIATED FROM THE CAVITY  
THROUGH THE COUPLER (AVERAGE ON ...)

LOADED Q

$$Q_L = \omega_0 \frac{W}{P_{CAV} + P_{EXT}}$$

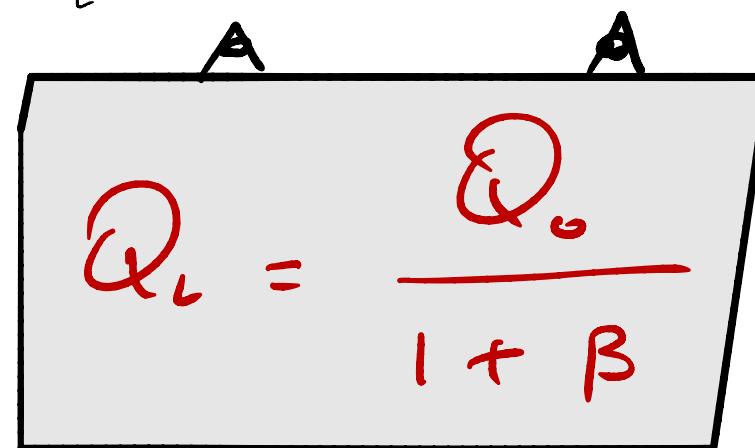
EXTERNAL Q

$$Q_{EXT} = Q_E = \omega_0 \frac{W}{P_{EXT}}$$

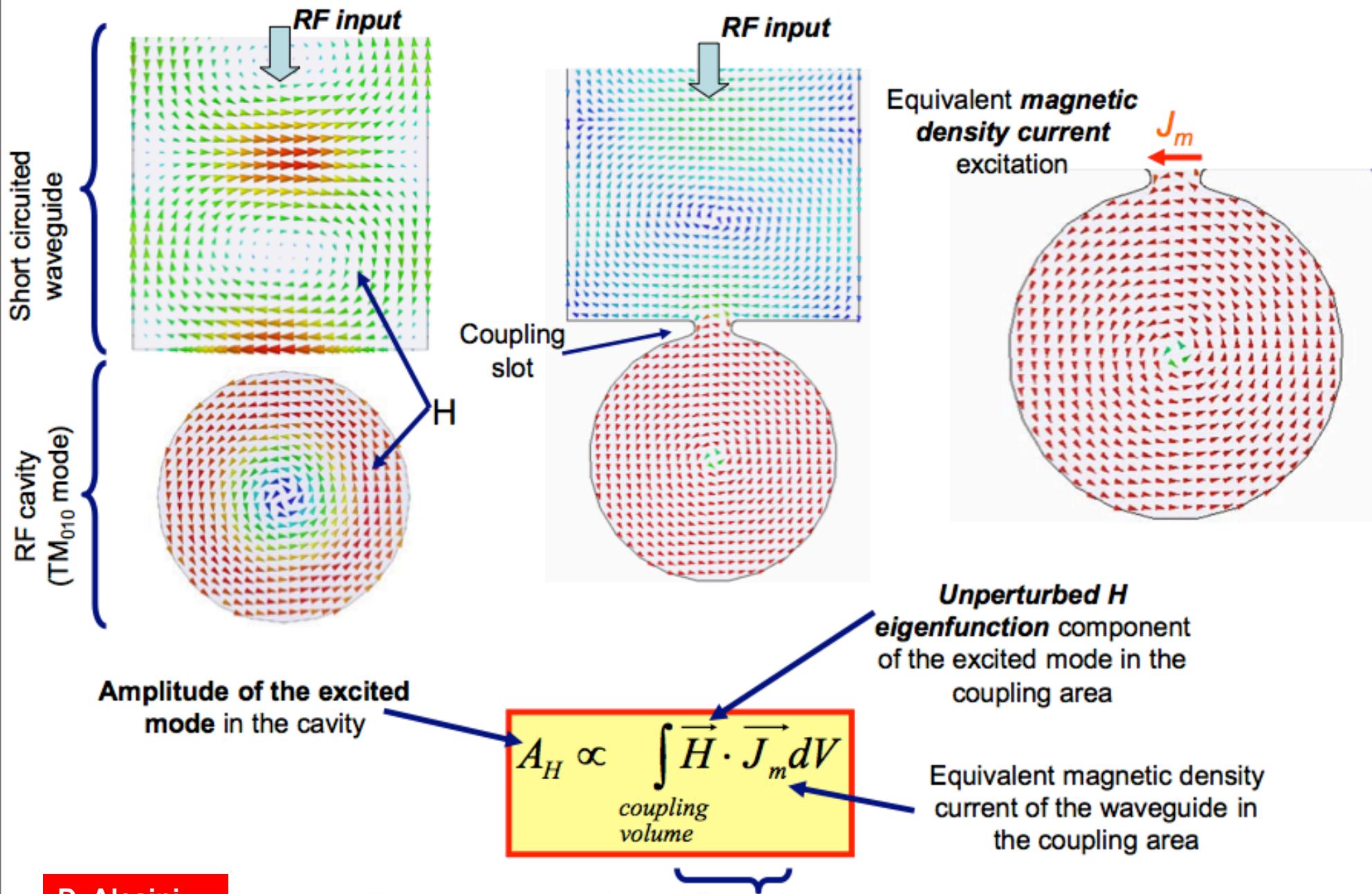
COUPLING B

$$\beta = \frac{P_{EXT}}{P_{CAV}} = \frac{Q_o}{Q_E} = \frac{R}{n^2 Z_o}$$

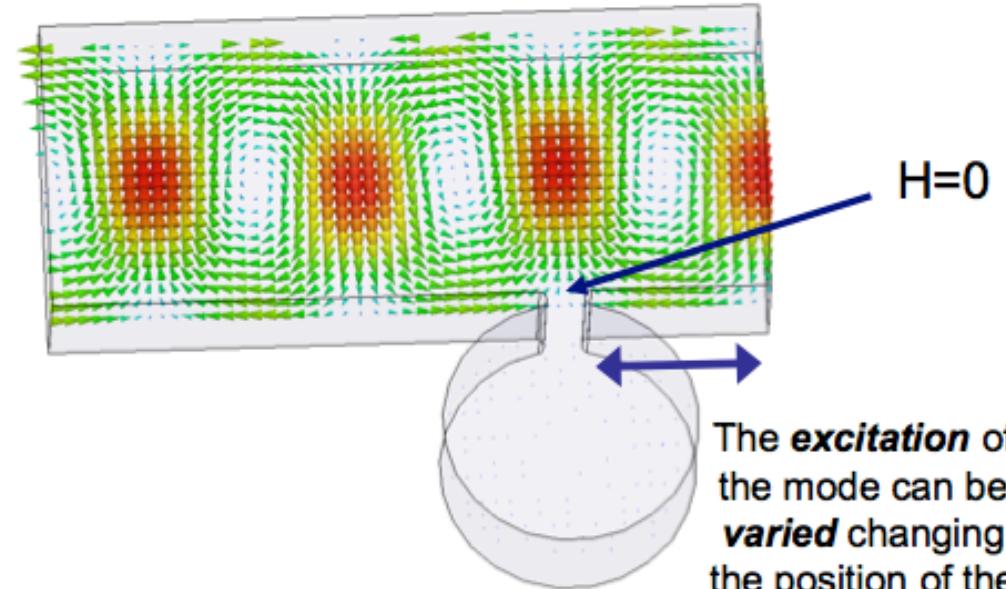
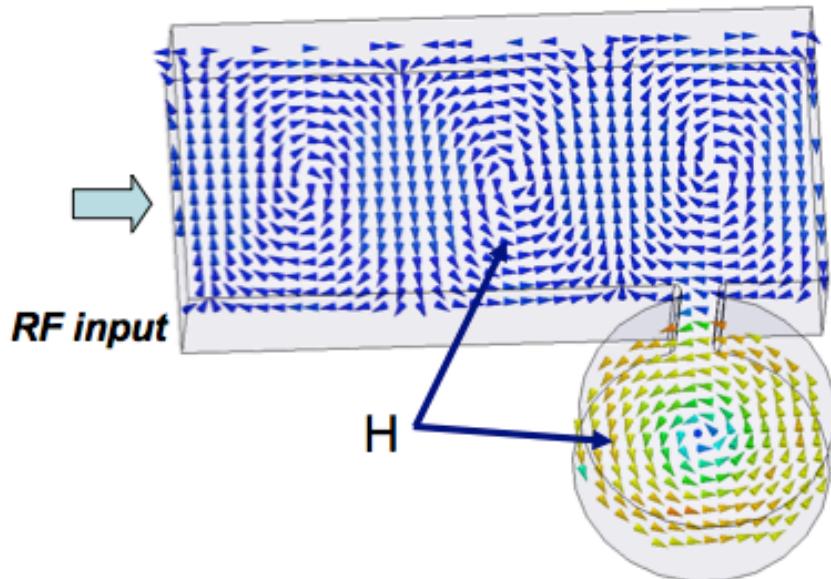
$$\frac{1}{Q_L} = \frac{1}{Q_E} + \frac{1}{Q_o}$$



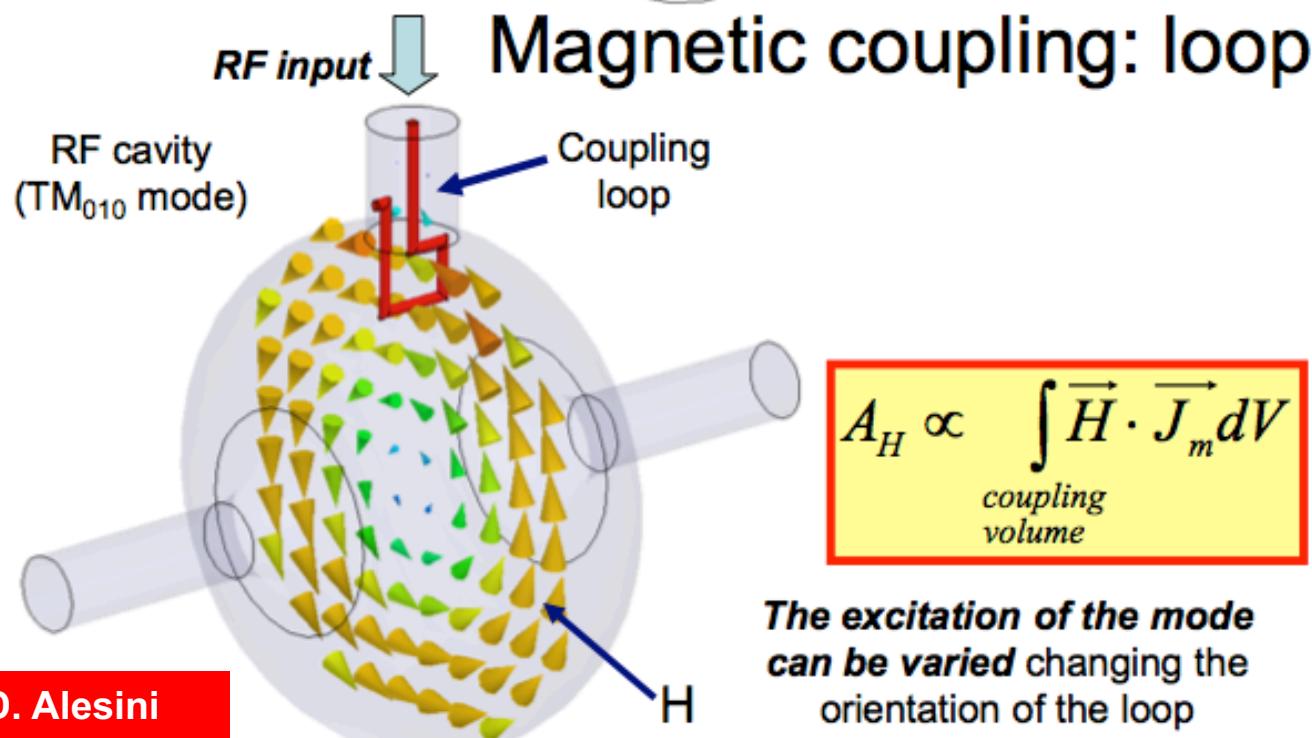
# Magnetic coupling: slots on waveguides



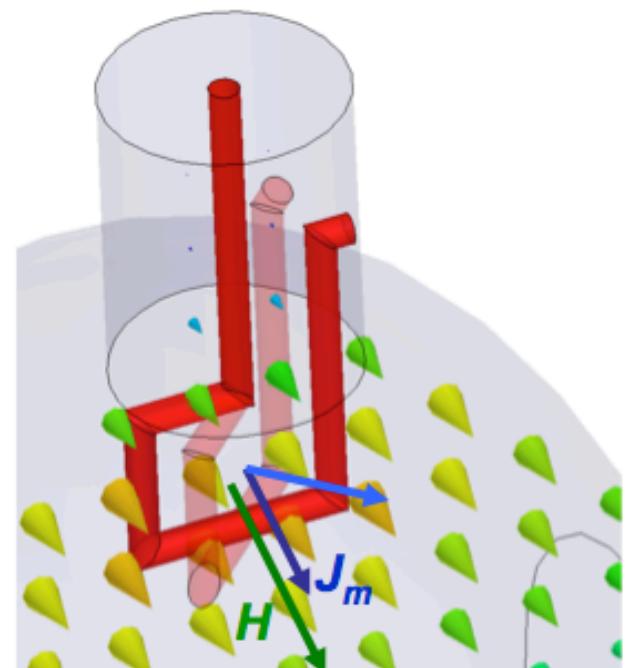
# Magnetic coupling: longitudinal slots on waveguides



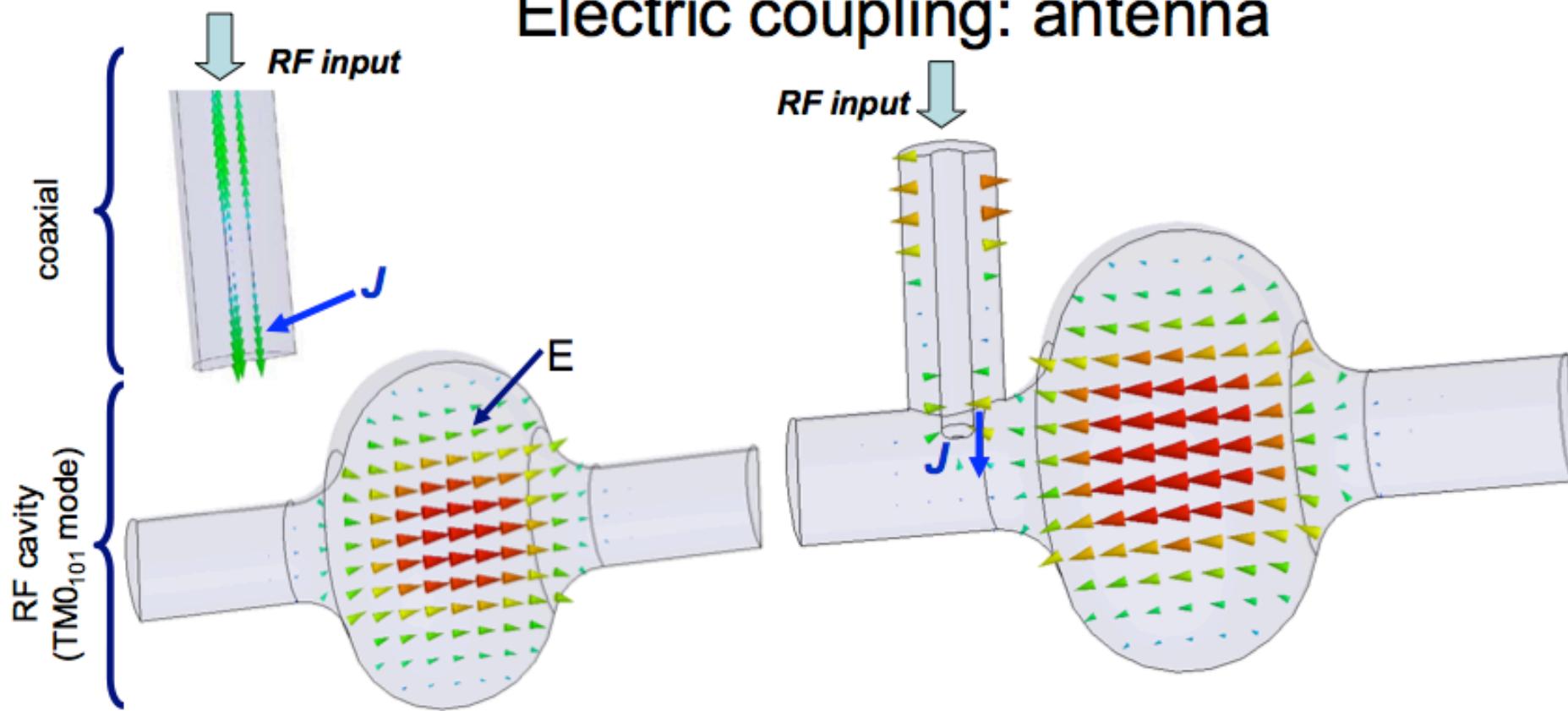
The **excitation** of the mode can be **varied** changing the position of the short circuit plane



The **excitation of the mode** can be **varied** changing the orientation of the loop



# Electric coupling: antenna



Amplitude of the excited mode in the cavity

$$A_E \propto \int_{\text{coupling volume}} \vec{E} \cdot \vec{J} dV$$

Unperturbed  $E$  eigenfunction component of the excited mode in the coupling area

Equivalent electric density current of the waveguide in the coupling area

Both  $E$  and  $J$  should be different from zero and non-orthogonal in order to have excitation of the mode

## MEASUREMENT OF CAVITY PARAMETERS

$$Z_{\text{CAV}} = \frac{\beta Z_0}{1 + j Q_0 \delta}$$

$$\begin{matrix} S_{\text{CAV}} \\ (S_{11}) \end{matrix}$$

$$= \frac{Z_{\text{CAV}} - Z_0}{Z_{\text{CAV}} + Z_0} = \frac{\beta - 1 - j Q_0 \delta}{\beta + 1 + j Q_0 \delta}$$

$$|\beta_{\text{CAV}}| = \sqrt{\frac{\left(\frac{\beta-1}{\beta+1}\right)^2 + (Q_L \delta)^2}{1 + (Q_L \delta)^2}}$$

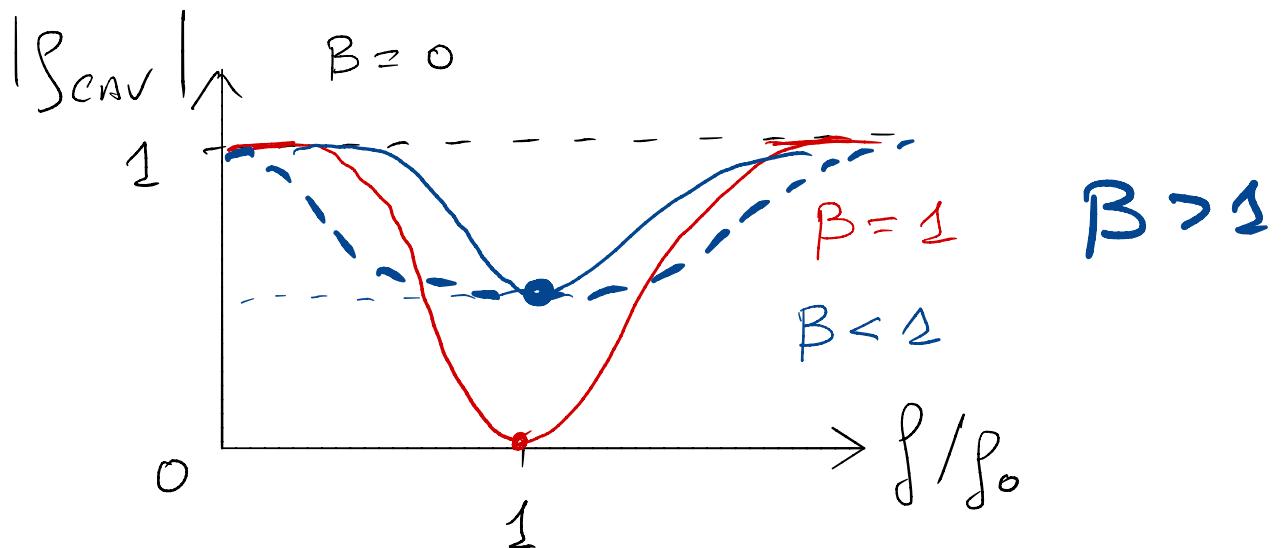
$$Q_L = \frac{Q_0}{1 + \beta}$$

$$|\beta_{\text{CAV}}| = \left| \frac{\beta-1}{\beta+1} \right|$$

$\delta = \delta_0 \quad \delta = 0$

$$\angle \beta_{\text{CAV}} = -\tan^{-1} \left[ \frac{2\beta Q_0 \delta}{\beta^2 - 1 - (Q_0 \delta)^2} \right] \approx \frac{2\beta}{1 - \beta^2} Q_0 \delta$$

$$\approx \frac{2\beta}{1 - \beta^2} Q_0 \frac{2(f - f_0)}{f_0} \quad \begin{matrix} Q_0 \delta \ll 1 \\ \Rightarrow \text{LINE} \end{matrix}$$



$$\left| \frac{\beta - 1}{\beta + 1} \right| = \begin{cases} \beta = 2 & \rightarrow \frac{1}{3} \\ \beta = 0,5 & \rightarrow \frac{1}{3} \end{cases}$$

$\beta < 1$

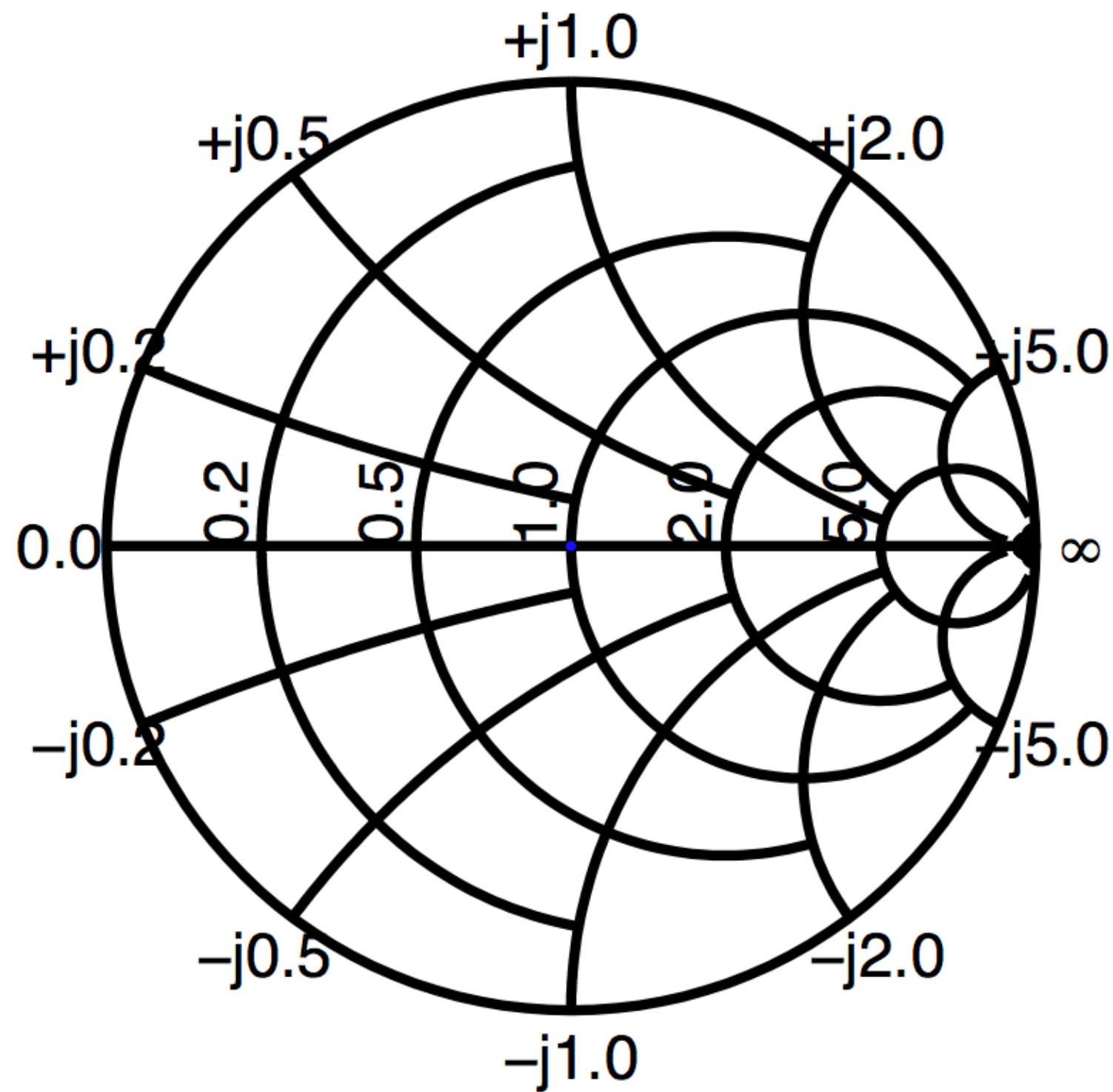
UNDER - COUPLING

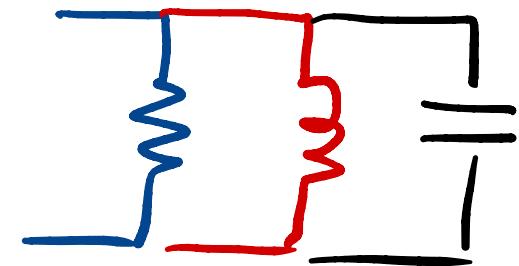
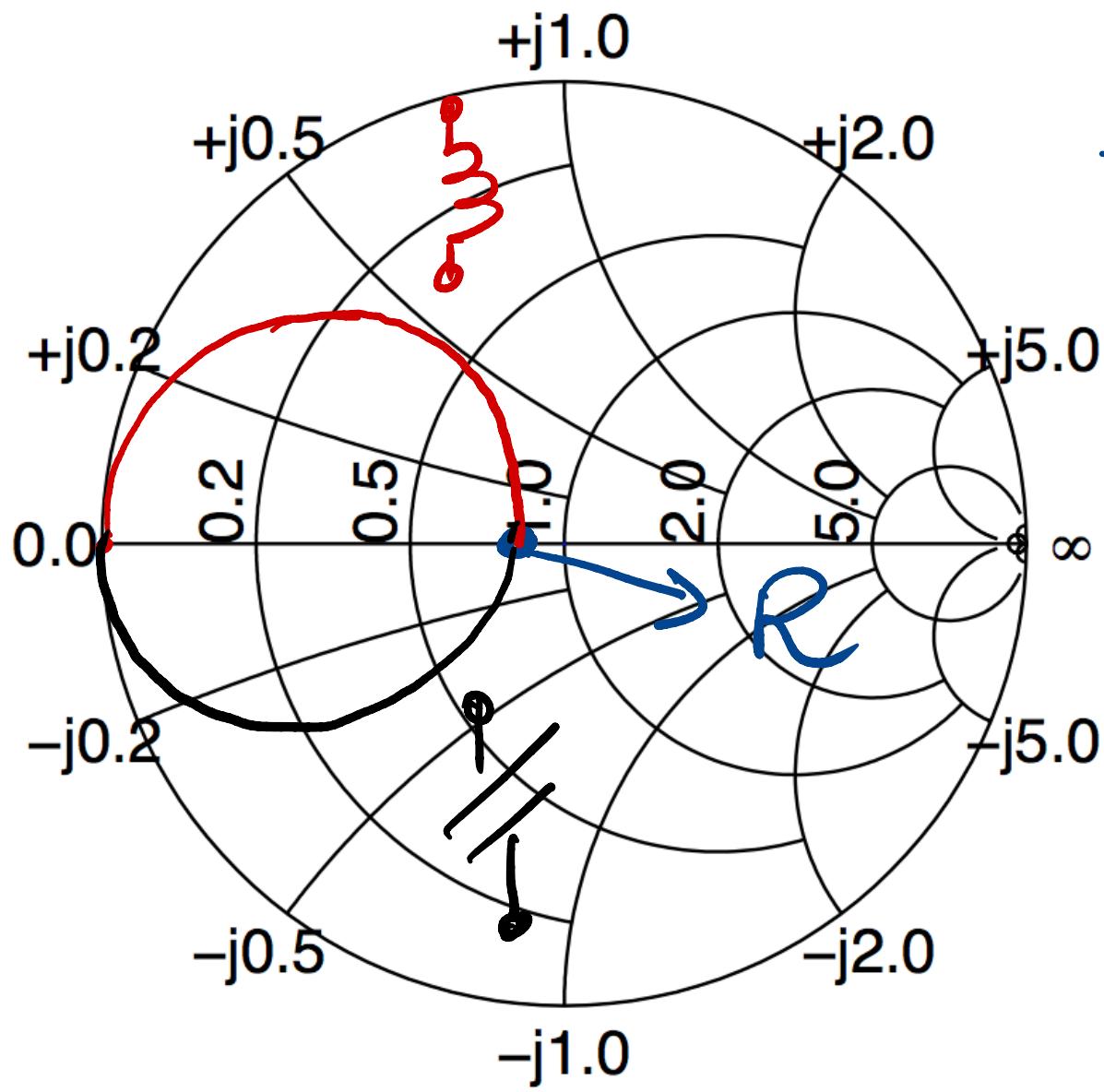
$\beta = 1$

CRITICAL COUPLING

$\beta > 1$

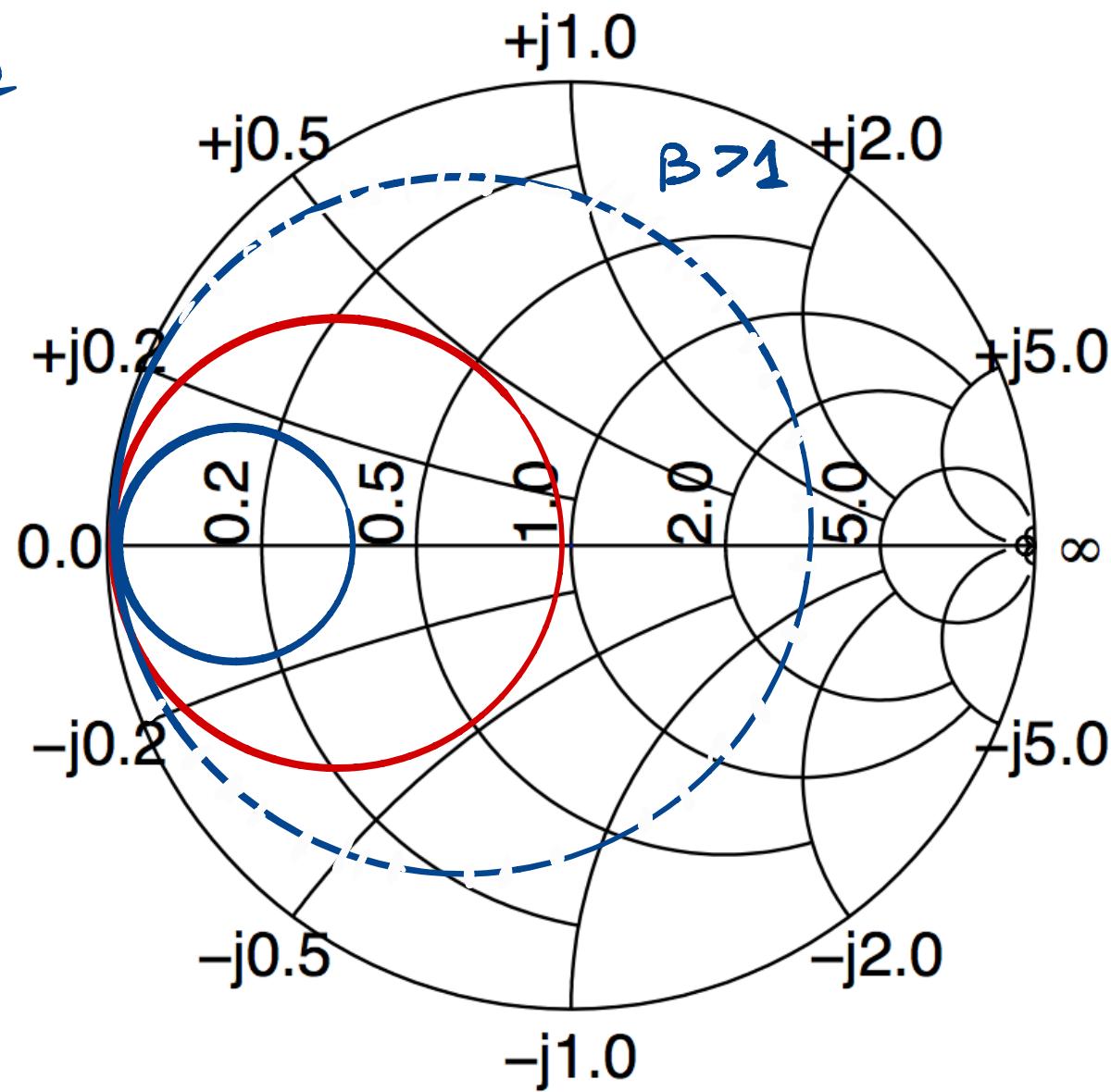
OVER - COUPLING





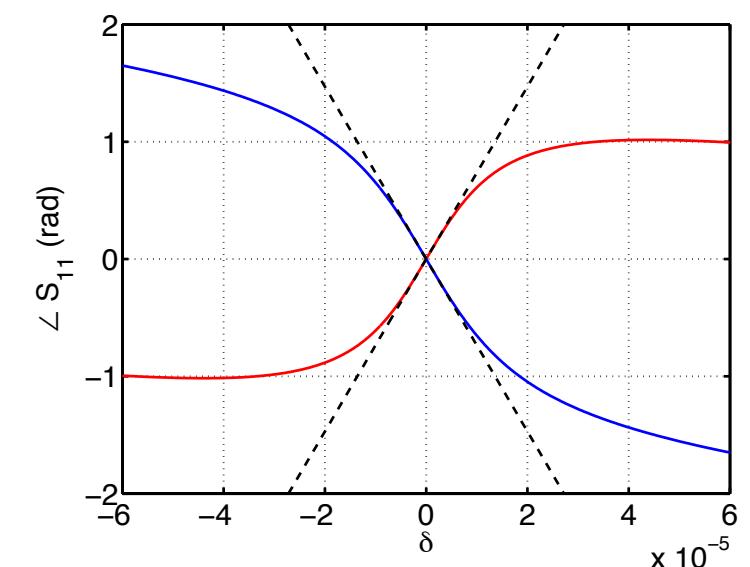
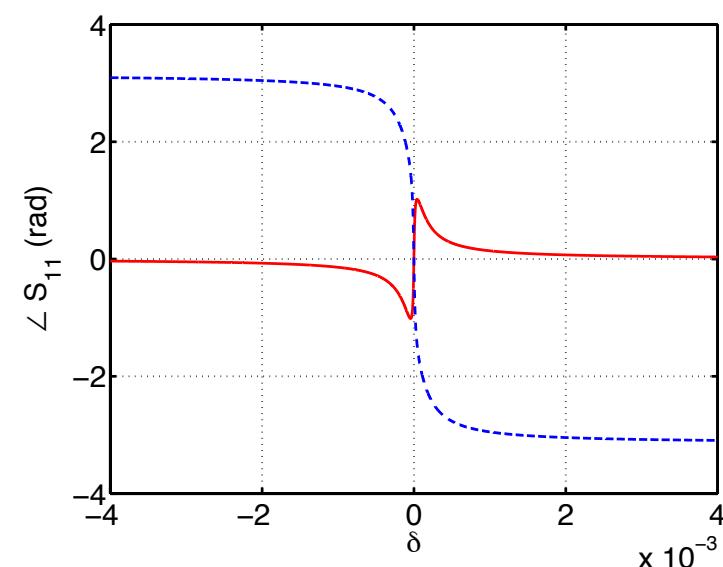
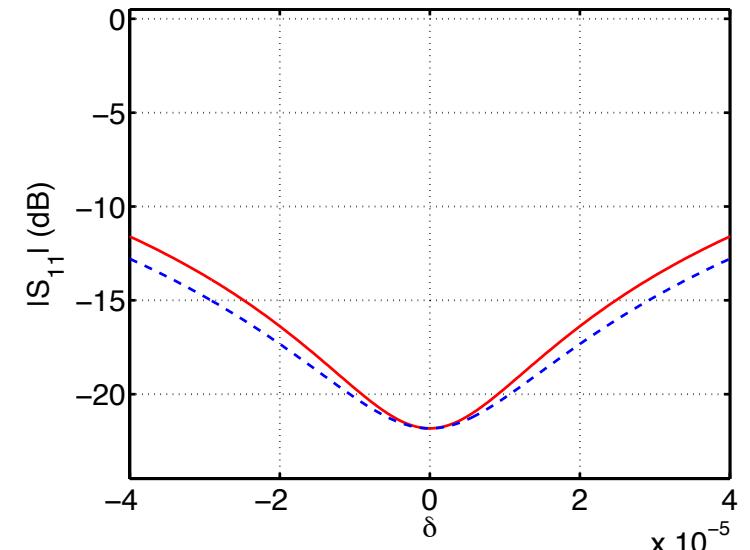
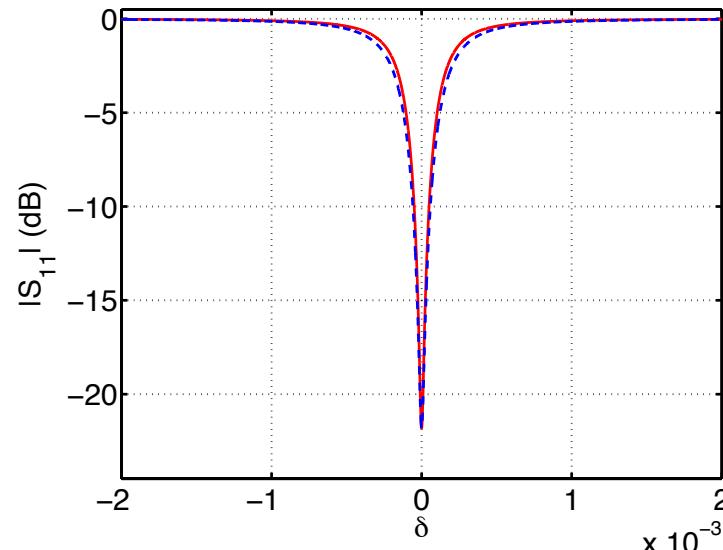
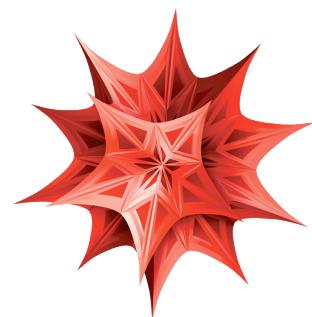
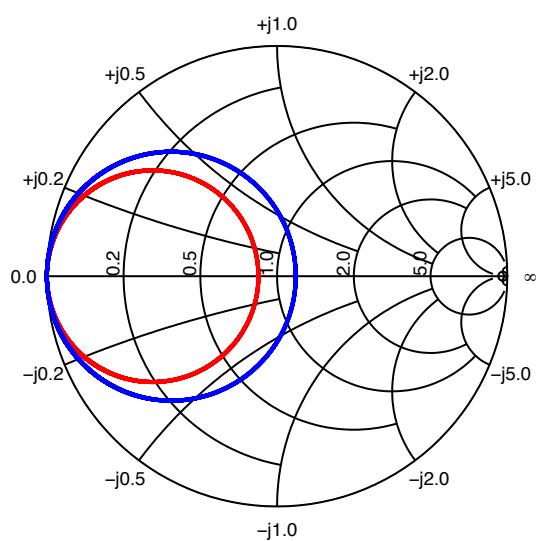
$$\beta = 1$$

$$\beta < 1$$

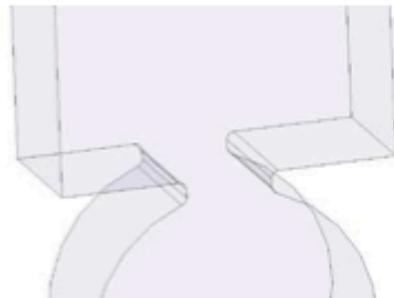


$$\beta = \frac{R}{n^2 \epsilon_0}$$

## Undercoupling versus overcoupling

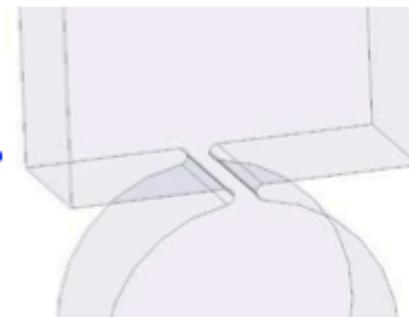
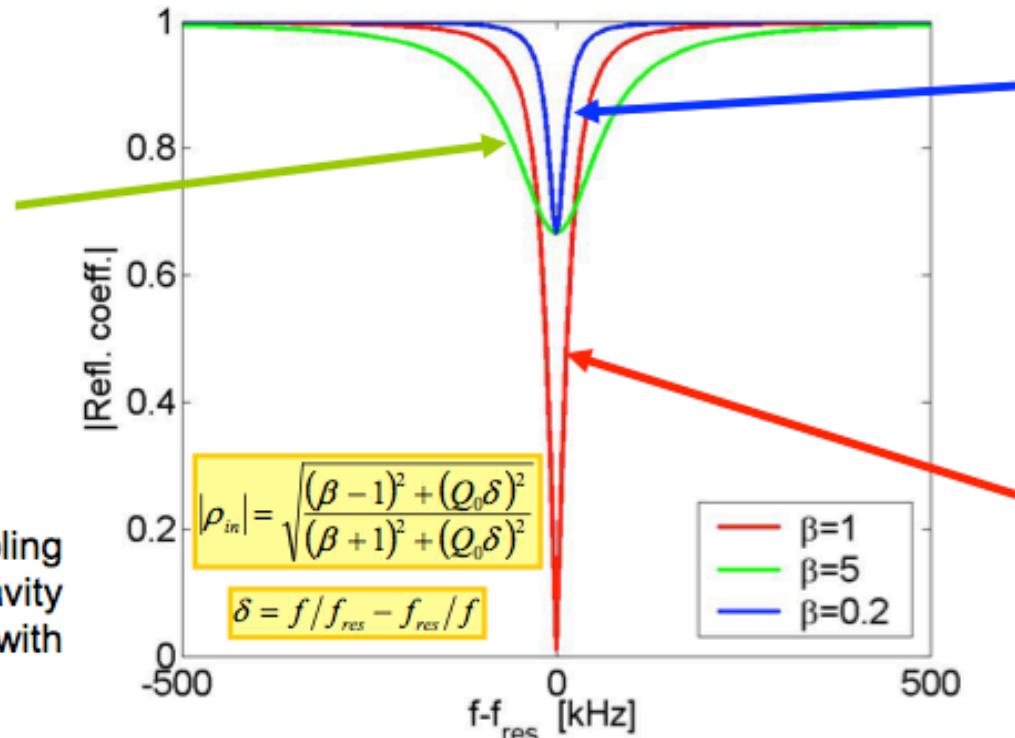


# Coupling coefficient

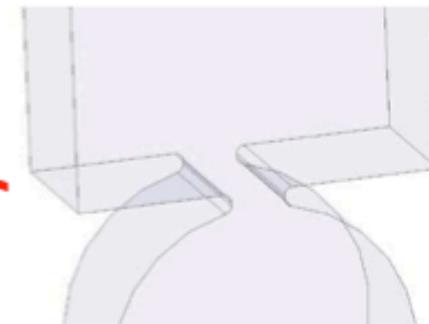


$\beta > 1$  over coupling

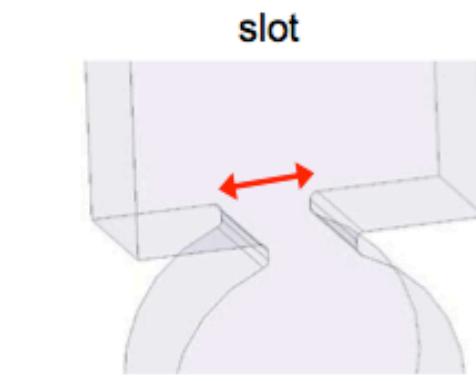
The choice of the coupling depends on the cavity operational condition with beam.



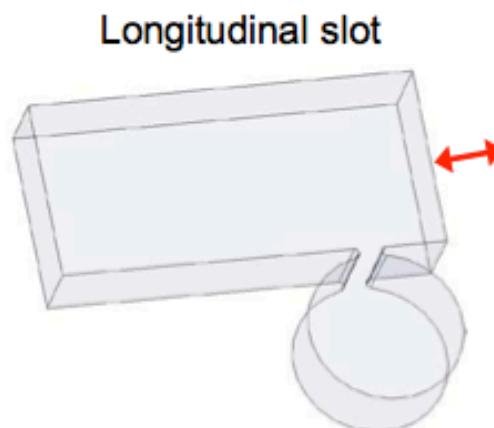
$\beta < 1$  under coupling



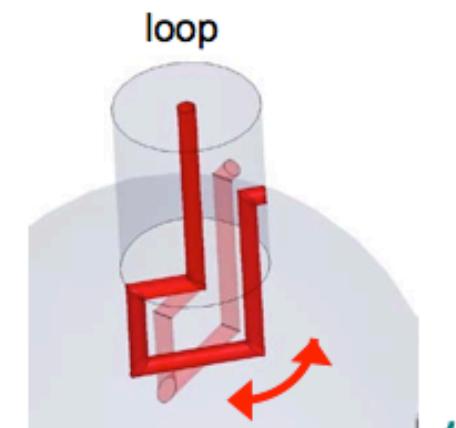
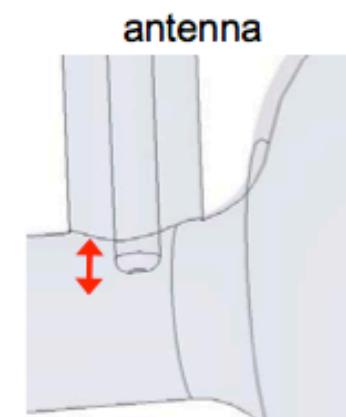
$\beta = 1$  critical coupling



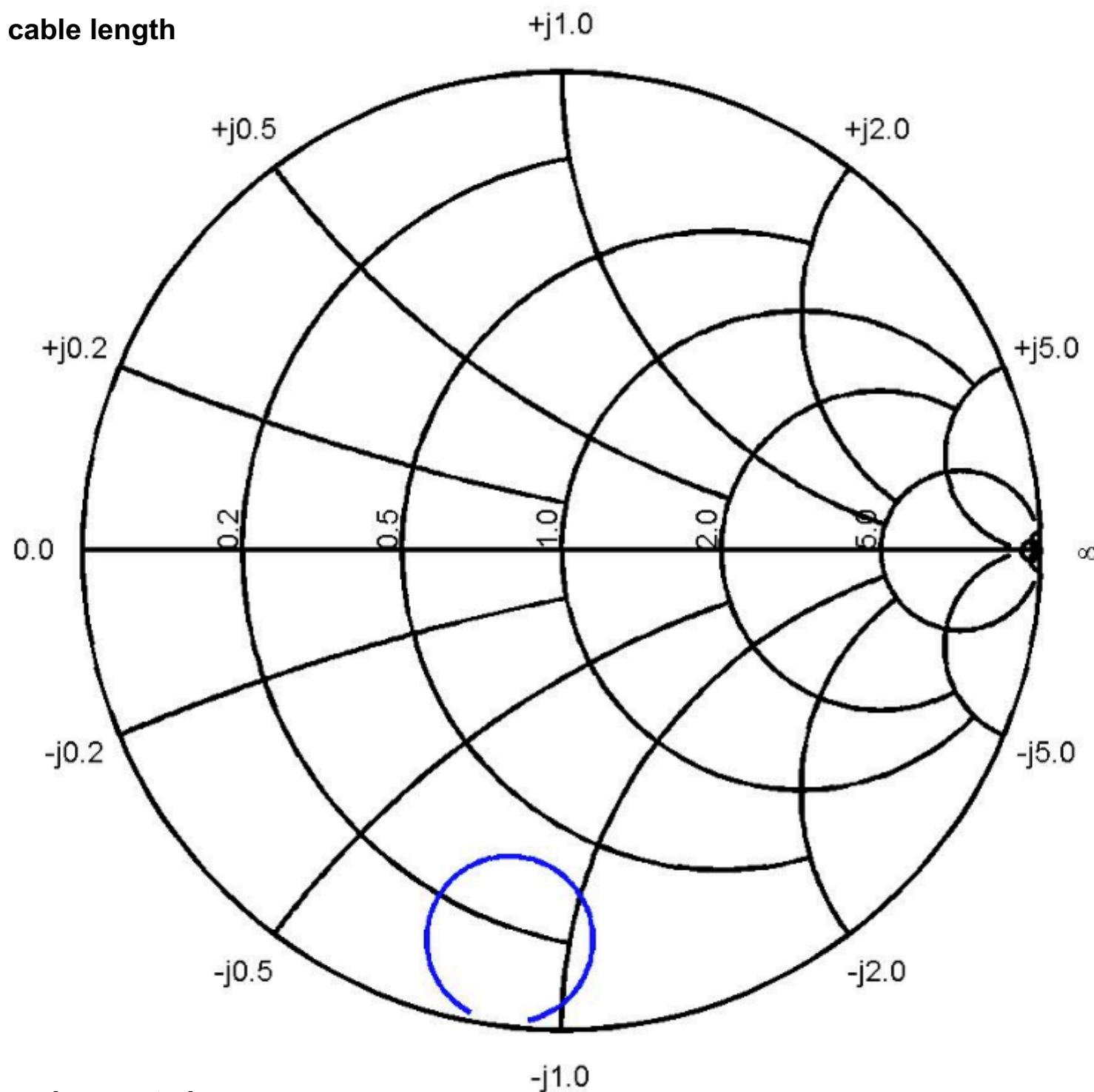
The coupling is fixed once we have construct the cavity



It is possible to **change the coupling** changing the position of the short circuit plane, the antenna penetration or the loop orientation

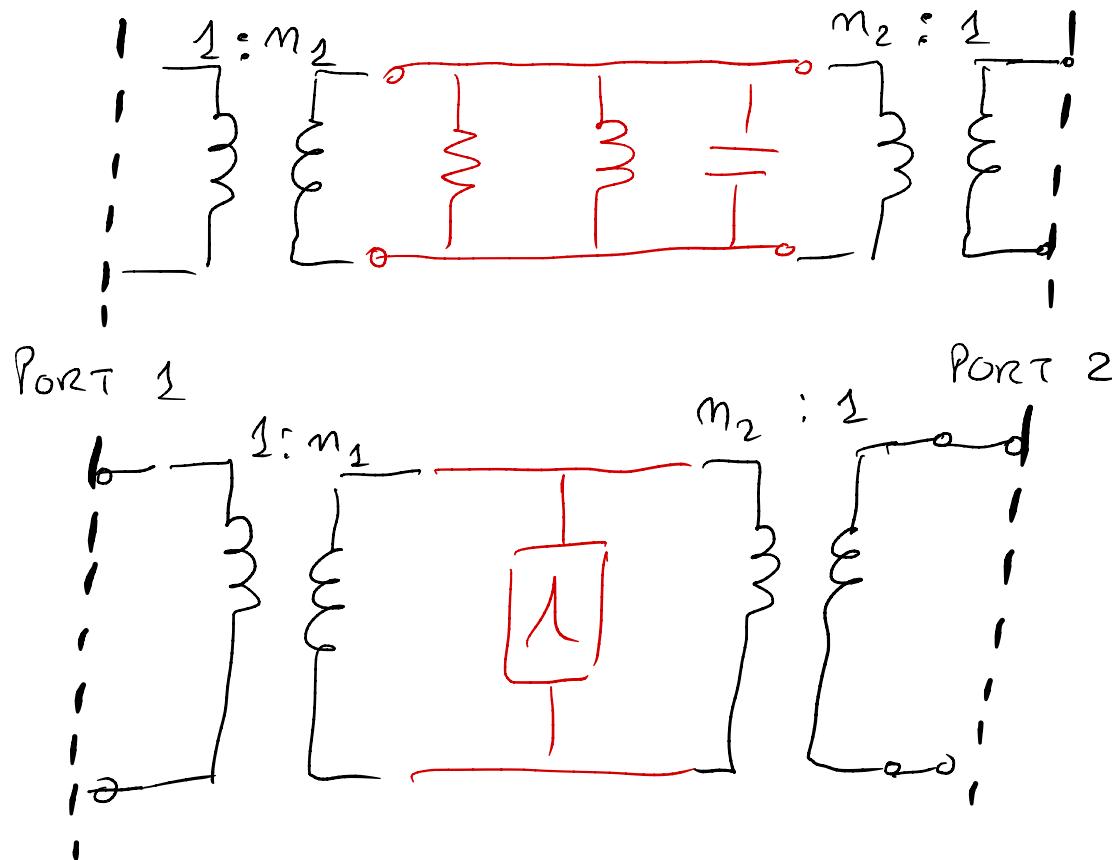


## Effect of the cable length



.. the coupling does not change

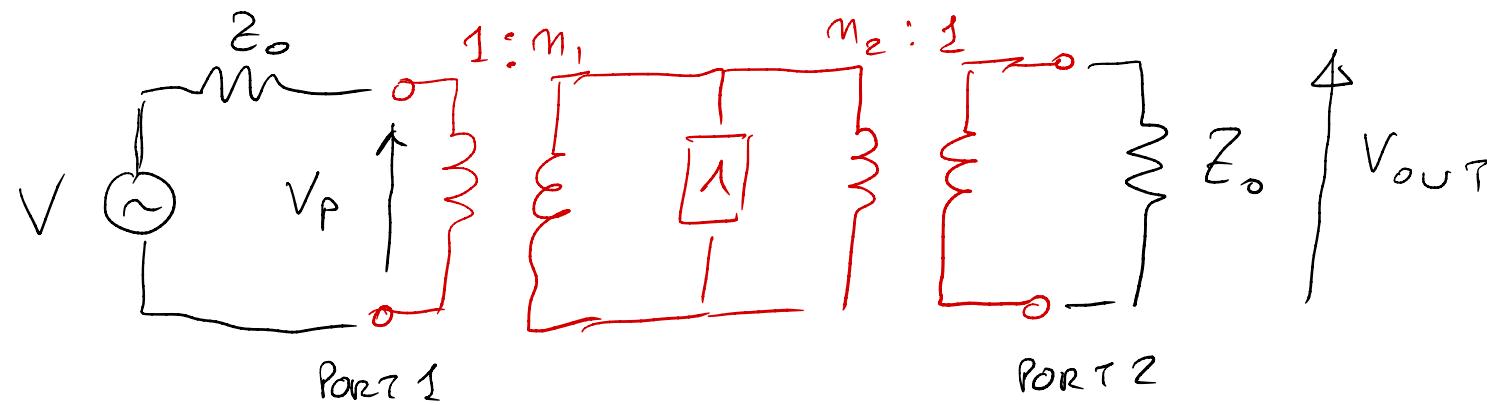
# CAVITY TRANSMISSION MEASUREMENT



$$Z = \frac{R}{1 + jQ_0\delta}$$

COMPUTATION OF  $S_{21}$

# COMPUTATION OF $S_{21}$

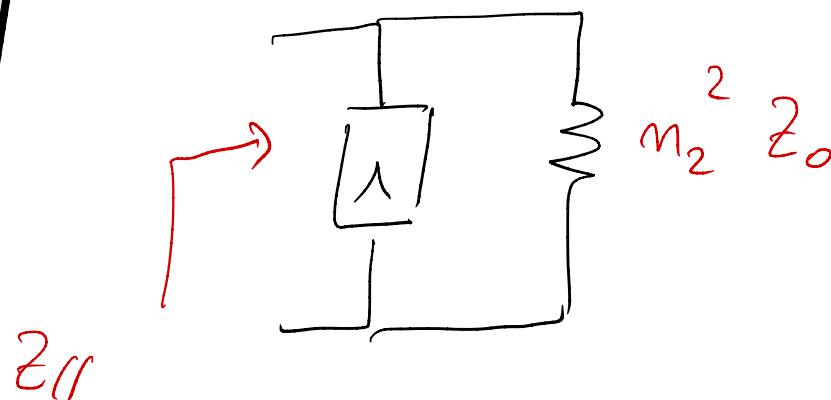


$$S_{21} = \frac{V_{\text{OUT}}}{V_{\text{INC}}}$$

$$\left. \begin{aligned} V_p &= (1 + S_{11}) V_{\text{INC}} \\ V_{\text{OUT}} &= V_p \frac{n_1}{n_2} \end{aligned} \right\} S_{21} = \frac{V_{\text{OUT}}}{V_{\text{INC}}} = (1 + S_{11}) \frac{n_1}{n_2}$$

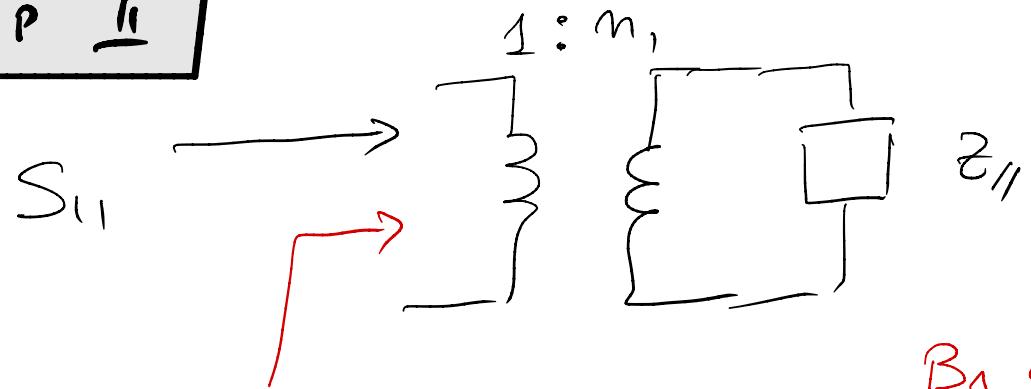
**STEP I**

$$\frac{R}{n_2^2} = \beta_2 Z_0$$



$$\begin{aligned} Z_{II} &= \frac{\frac{R}{1 + jQ_0\delta} n_2^2 Z_0}{\frac{R}{1 + jQ_0\delta} + n_2^2 Z_0} \\ &= \frac{R}{1 + \beta_2 + jQ_0\delta} \end{aligned}$$

## STEP II



$$\frac{R}{n_1^2} = \beta_1 Z_0$$

$$Z_I = \frac{Z_{11}}{n_1^2} = \frac{\beta_1 Z_0}{1 + \beta_2 + jQ_0\delta}$$

$$S_{11} = \frac{Z_I/Z_0 - 1}{Z_I/Z_0 + 1} = \frac{\beta_1 - 1 - jQ_0\delta}{\beta_1 + 1 + \beta_2 + jQ_0\delta}$$

$$1 + S_{11} = \frac{2\beta_1}{1 + \beta_1 + \beta_2 + jQ_0\delta} \quad \left( = \frac{2 Z_I/Z_0}{Z_I/Z_0 + 1} \right)$$

$$S_{21} = (1 + S_{11}) \frac{n_1}{n_2} =$$

$$2 \sqrt{\beta_1 \beta_2}$$

=

$$1 + \beta_1 + \beta_2 + j Q_o \delta$$

$$S_{11} = \frac{\beta_1 - 1 - \beta_2 - j Q_o \delta}{1 + \beta_1 + \beta_2 + j Q_o \delta} @ \text{PORT 2}$$

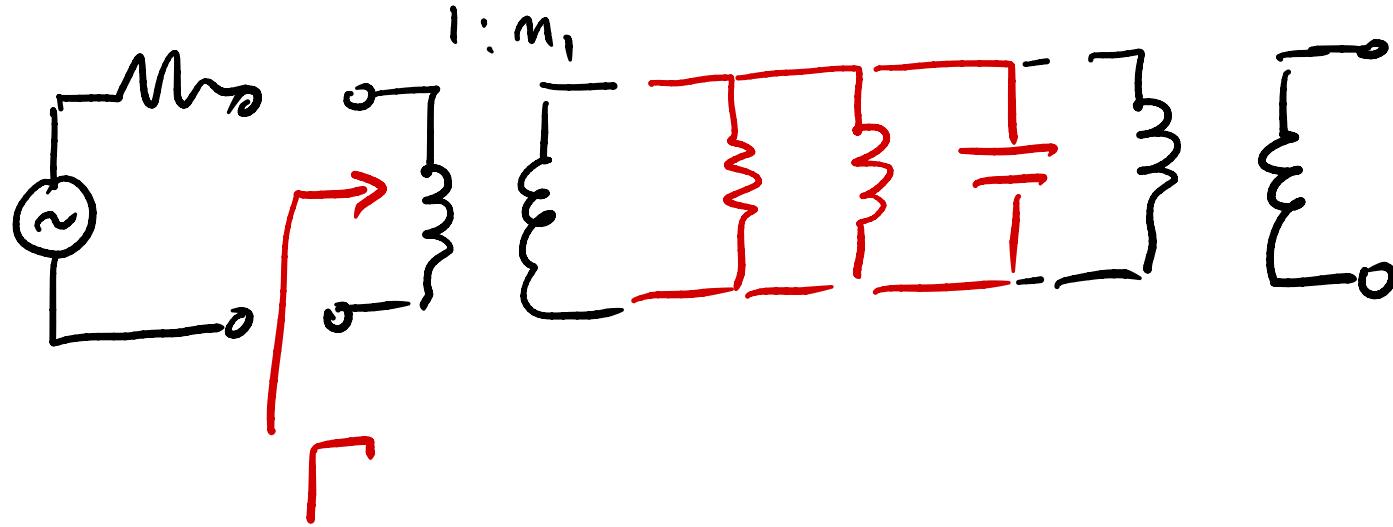
$$S_{22} = \frac{\beta_2 - 1 - \beta_1 - j Q_o \delta}{1 + \beta_1 + \beta_2 + j Q_o \delta} @ \text{PORT 1}$$

$$\left( \begin{aligned} \frac{\beta_1 n_1}{n_2} &= \frac{R/Z_o}{n_1^2} \frac{n_1}{n_2} = \\ &= \sqrt{\beta_1 \beta_2} \end{aligned} \right)$$

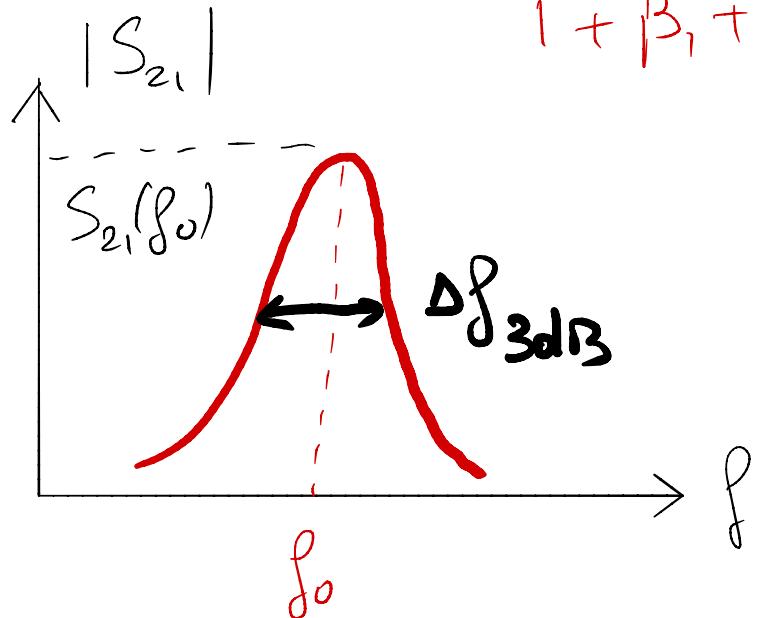
$$\beta_J = \frac{R/Z_o}{n_J}$$

$$\Gamma_1 = \frac{\beta_1 - 1 - j Q_o \delta}{1 + \beta_1 + j Q_o \delta} \quad \text{OPEN @ PORT 2}$$

$$\Gamma_2 = \frac{\beta_2 - 1 - j Q_o \delta}{1 + \beta_2 + j Q_o \delta} \quad \text{OPEN @ PORT 1}$$



$$S_{21} = \frac{\frac{2\sqrt{\beta_1\beta_2}}{1+\beta_1+\beta_2}}{1 + j \frac{Q_0}{1+\beta_1+\beta_2} \delta} = \frac{S_{21}(f_0)}{1 + j Q_L \delta}$$



$$Q_L = \frac{f_0}{\Delta f_{3dB}}$$

AUTOMATIC  
Q MEAS !!

- $\beta_1, \beta_2 \ll 1$        $Q_L \approx Q_0$        $S_{21}(f_0) \ll 1$

BUT STILL  
MEASURABLE

PRACTICAL WAY

- $\beta_1 = \beta_2 = \beta$       SYMMETRIC COUPLING

$$S_{21} = \frac{2\beta}{1 + j Q_L \delta}$$

PHASE OF  $S_{21}$

$$\angle S_{21} = -t_g^{-1}(Q_L \delta) \approx -Q_L \delta = -Q_L \frac{2(f-f_0)}{f_0}$$

$Q_L \delta \ll 1$

