S_{11} fitting routines for RF cavity resonances

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I. INTRODUCTION

A mode in a resonant cavity can be modeled as a resonant circuit; we will consider the parallel resonant circuit of Figure 1 where the transformer accounts for the circuit exciting the mode. In this modelization the impedance of the cavity is

$$Z_c(\omega) = \frac{R}{1 + jQ_0\delta} \quad \text{with} \quad Q_0 = R\sqrt{\frac{C}{L}}, \; \omega_0 = \frac{1}{\sqrt{LC}}, \; \delta = \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}. \tag{1}$$

The power dissipated in the cavity provided that V_c is the voltage at the ends of the parallel circuit is $P_{wall} = V_c^2/2R$ and it is related to the energy stored in the cavity U since $Q = \omega_0 U/P_d$. The impedance of the cavity, as seen from the transformer end, is

$$Z(\omega) = \frac{R/n^2}{1 + jQ_0\delta} = \frac{\beta Z_0}{1 + jQ_0\delta} \quad \text{since} \quad \beta Z_0 = \frac{R}{n^2}. \tag{2}$$

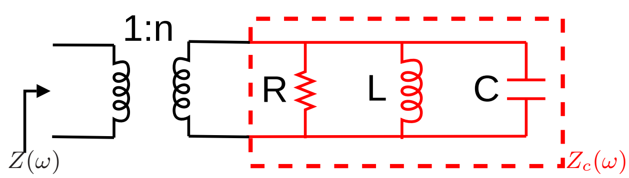

FIG. 1: Cavity parallel circuit [1].

The reflection coefficient measured at the coupling antenna/waveguide is

$$S_{11} = \frac{Z(\omega) - Z_0}{Z(\omega) + Z_0} = \frac{\beta - 1 - jQ_0\delta}{\beta + 1 + jQ_0\delta} = \frac{\beta^2 - 1 - (Q_0\delta)^2}{(\beta+1)^2 + (Q_0\delta)^2} - j\frac{2\beta Q_0\delta}{(\beta+1)^2 + (Q_0\delta)^2}. \tag{3}$$

The amplitude of the reflection coefficient is

$$|S_{11}| = \sqrt{\frac{(\beta-1)^2 + (Q_0\delta)^2}{(\beta+1)^2 + (Q_0\delta)^2}}. \tag{4}$$

From the amplitude alone, one can not distinguish under-coupled from over-coupled resonance since the transformation $\beta \to 1/\beta$ and $Q_0 \to Q_0/\beta$ results in the same $|S_{11}|$. Therefore the phase (or the Smith charth plot) must be checked; the phase reads:

$$\angle S_{11} = \arctan\left(\frac{Q_0\delta}{1-\beta}\right) - \arctan\left(\frac{Q_0\delta}{1+\beta}\right) = -\arctan\left(\frac{2\beta Q_0\delta}{\beta^2 - 1 - (Q_0\delta)^2}\right), \tag{5}$$

where the first relation prevents from fictious phase jumps. Around the resonace frequency, $\angle S_{11}$ can be approximated as

$$\angle S_{11} \approx \frac{2\beta}{1-\beta^2} Q_0\delta = \frac{2\beta}{1-\beta} Q_L\delta \quad \text{if} \quad \left|\frac{Q_0\delta}{1-\beta}\right| \ll 1. \tag{6}$$

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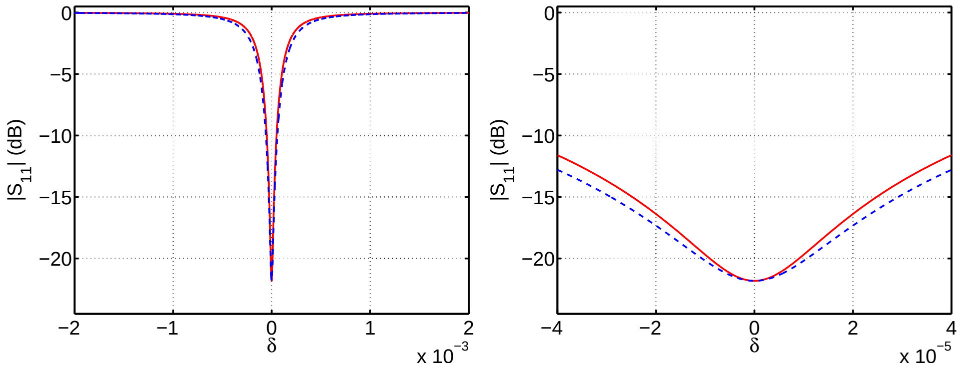

FIG. 2: Magnitude of the S_{11} according to Eq. 4 as a function of the normalised frequency shift δ in a wide range (left picture) and in a narrow range (right picture) around the resonance $\delta = 0$. The red line is obtained with $\beta = 0.85$ and $Q_0 = 12000$ (under-coupling), while the blue line is for $\beta = 1/0.85$ and the same Q_0 (over-coupling).

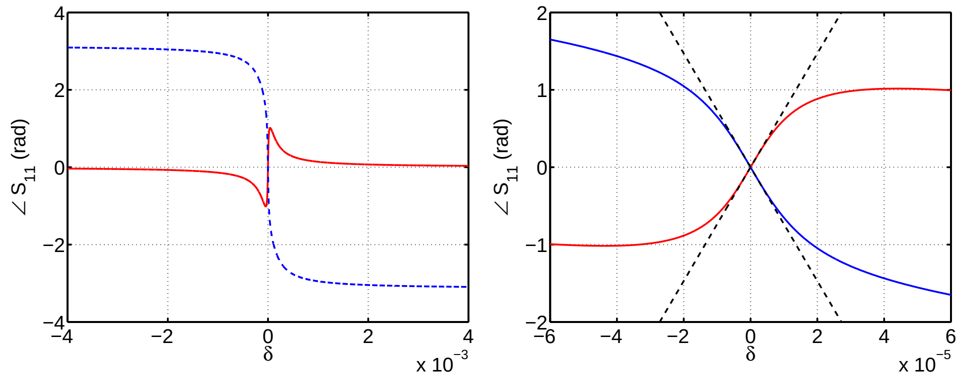

FIG. 3: Phase of the S_{11} according to Eq. 5 as a function of the normalised frequency shift δ in a wide range (left picture) and in a narrow range (right picture) around the resonance $\delta = 0$. The red line is obtained with $\beta = 0.85$ and $Q_0 = 12000$ (under-coupling), while the blue line is for $\beta = 1/0.85$ and the same Q_0 (over-coupling). The dashed black line are the first order approximation around the resonance frequency according to Eq. 6.

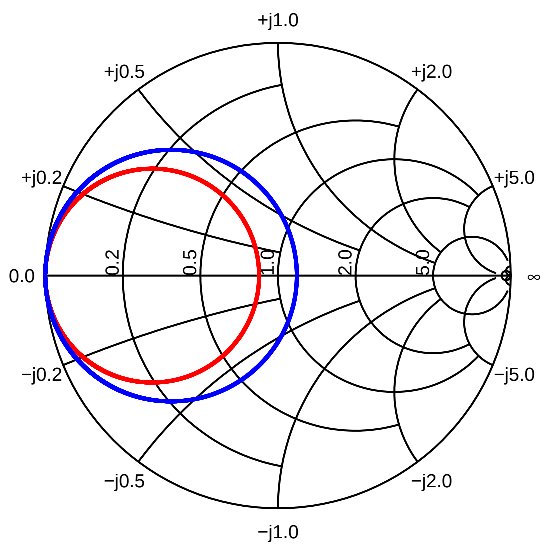

FIG. 4: Smith chart of the of the S_{11} according to Eq. 3 for f_0=2.856 GHz $Q_0 = 12000$ and $\beta = 0.85$ (1/0.85) for the red (blue) line, i.e. under-coupling (over-coupling) condition.

II. ROUTINES DESCRIPTION

The model used in the fit is the single resonance reflection coeficinent of Eq. 3 considering a constant attenuation α:

$$|S_{11}| = \alpha\sqrt{\frac{(\beta-1)^2+(Q_0\delta)^2}{(\beta+1)^2+(Q_0\delta)^2}}. \quad \text{with} \quad \alpha\in[0,1]. \tag{7}$$

As an estimation of the quality of the fit, we could quadratically sum the difference between the data $\widetilde{S_{11}}$ and its estimation S_{11}, i.e.

$$\sigma_{||} = \sqrt{\frac{\sum\left(|S_{11}|-\left|\widetilde{S_{11}}\right|\right)^2}{N-N_{par}}} \quad \text{and} \quad \sigma_{\angle} = \sqrt{\frac{\sum\left(\angle S_{11}-\angle\widetilde{S_{11}}\right)^2}{N-N_{par}}}, \tag{8}$$

where the sum is weighted for the number of effective degree of freedoms that is the difference between the number of data points and the number of fit parameters N_{par}.

- Interpolation from Eq. 7 keeping f_0, β, Q_0 as free parameters of the fit. The attenuation α is given. Results in the following figures are drawn with magenta lines.
  MATLAB routine: S11fit.m.

- Interpolation from Eq. 7 keeping β, Q_0 as free parameters of the fit. The resonance frequency is chosen as the minimum of the $|S_{11}|$ and its unceratinty as the frequency sample separation divided by $\sqrt{12}$. The attenuation α is given. Results in the following figures are drawn with blue lines.
  MATLAB routine: S11fit_delta.m.

- Interpolation from Eq. 7 keeping β, Q_0 and α as free parameters of the fit. The resonance frequency is chosen as the minimum of the $|S_{11}|$ and its unceratinty as the frequency sample separation divided by $\sqrt{12}$. Results in the following figures are drawn with green lines.
  MATLAB routine: S11fit_delta_attentation.m.

- Interpolation from Eq. 7 keeping f_0, β, Q_0 and α as free parameters of the fit. Results in the following figures are drawn with black lines.
  MATLAB routine: S11fit_attentation.m.

[1] A. Gallo, *Beam Loading and Low-Level RF Control in Storage Rings*, Cern Accelerator School, Trieste (Italy) 2005.