Guide to the expression of uncertainty in measurement

The uncertainty of a measurement is a parameter associated with the result of the measurement, that characterises the dispersion of the values that could reasonably be attributed to the quantity being measured.

The Guide and the Vocabulary …


BIPM

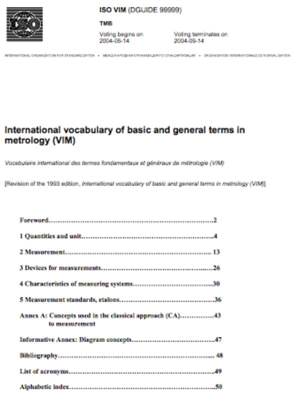
JCGM 100:2008
GUM 1995 with minor corrections

Evaluation of measurement data — Guide to the expression of uncertainty in measurement

Évaluation des données de mesure — Guide pour l'expression de l'incertitude de mesure

First edition September 2008

© JCGM 2008


ISO VIM (DGUIDE 99999)

TMB

Voting begins on
2004-05-14

Voting terminates on
2004-09-14

International vocabulary of basic and general terms in metrology (VIM)

Vocabulaire international des termes fondamentaux et généraux de métrologie (VIM)

[Revision of the 1993 edition, *International vocabulary of basic and general terms in metrology (VIM)*]

Basic concepts and definitions

- *The uncertainty (of measurement) is a parameter, associated with the result of a measurement, that characterises the dispersion of the values that could reasonably be attributed to the quantity to be measured.*
- *The method for evaluating the uncertainty is universal. The quantity expressing the uncertainty is internally consistent and transferable.*
- *The error of the result of a measurement and the true value of the measurand are UNKNOWABLE.*

Notations and symbols

$Y = f\left(X_1, \ldots, X_N\right)$ *y, x_k estimates of Y, X_k.*

Standard uncertainty: input quantities. $u(x_k)$

Combined uncertainty: output quantity. $u_c(y)$

Law of propagation

Expanded uncertainty: error bars. U_p

- *Uncertainty components may be evaluated in two different ways: Type A (statistical analysis) and Type B (any other means) evaluation.*

Type A evaluation of standard uncertainty

(single input quantity)

- *Statistical analysis of series of observations (frequency distribution).*

$$\overline{q} = \frac{1}{N}\sum_{k=1}^{N} q_k \qquad s(q_k) = \sqrt{\frac{1}{(N-1)}\sum_{k=1}^{N}(q_k - \overline{q})^2}$$

measurement result

N independent observations

uncertainty

$$u(q) = s(\overline{q}) = \frac{s(q_k)}{\sqrt{N}}$$

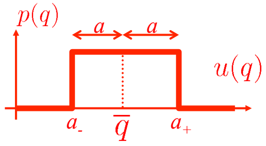
experimental standard deviation of the mean

Type B evaluation of standard uncertainty

(single input quantity)

- *Any means other than statistical analysis of series of observations (“a priori” distribution).*
- *Scientific judgement based on all available information such as: previous measured data, general knowledge of material properties ...*

No specific knowledge (upper and lower bounds).



$$u(q) = \frac{a}{\sqrt{3}}$$

Some knowledge (e.g. quoted uncertainty): most often normal distribution.

Type A evaluation vs Type B evaluation

- *All the uncertainty components are of the same nature. They are quantified by standard deviations (Type A) or variances (Type B); both types of evaluation are based on probability distributions.*
- *What is the “uncertainty on the uncertainty” (due to limited sampling)?*

N independent observations

Normal variables (ideal case).

N	**2**	**3**	**4**	**5**	**10**	**20**	**30**	**50**
$\frac{\sigma[s(\overline{q})]}{\sigma(\overline{q})}$ (%)	76	52	42	36	24	16	13	10

$\approx 1/\sqrt{2(N-1)}$

- *Repeated measurements are anyway needed to acquire knowledge of the experimental set-up.*

Combined uncertainty (law of propagation)

$$Y = f(x_1, \ldots, x_N) \qquad u_c(y) = \sqrt{\sum_{k=1}^{N}\left(\frac{\partial f}{\partial x_k}\right)^2 u^2(x_k)}$$

No correlated variables

combined uncertainty

sensitivity coefficients

- *The sensitivity coefficients can be determined experimentally, measuring a change in Y produced by a change in a particular x_k (other x_i constant).*
- *Numerical evaluation.*

$$\frac{\partial f}{\partial x_k}u(x_k) = \frac{1}{2}\left[f(x_1, \ldots, x_k + u(x_k), \ldots, x_N) - f(x_1, \ldots, x_k - u(x_k), \ldots, x_N)\right]$$

- *Correlated variables and f significantly non-linear: see the “Guide”.*

Expanded uncertainty U (I)

- *U defines an interval (about the measurement result) that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to the measurand.*

combined uncertainty

$$U_p = k_p \; u_c(y) \implies y \pm U_p$$

coverage factor

p — *coverage probability*

p — *level of confidence of the interval*

- *There is no new information, but the previously available information is presented in another way.*

Expanded uncertainty U (II)

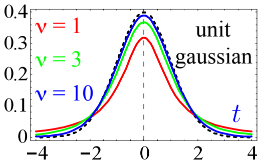
$$U_p = k_p \; u_c(y)$$

- *Central Limit Theorem.*
- *Often a practical approximation is using k_p from normal distributions.*

p (%)	68.3	90	95	99
k_p	1	1.64	1.96	2.58

- *Refined approach: Student's distribution and effective degrees of freedom.*

Student's t-distribution



$$t = (\bar{z} - \mu_z) \,/\, s(\bar{z})$$

Z normal variable

$$p(t,\nu) = \frac{1}{\sqrt{\pi\nu}} \frac{\Gamma\left[(\nu+1)/2\right]}{\Gamma(\nu/2)} \left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2}$$

- *If $\bar{z}$ and $s(\bar{z})$ are obtained by N independent observations, then the distribution of t is the t-distribution or Student's distribution with ν=N-1.*

$$\Pr\left[-t_p(\nu) \le t \le t_p(\nu)\right] = p$$

$$\Downarrow$$

$$\Pr\left[\bar{z} - t_p(\nu)s(\bar{z}) \le \mu_z \le \bar{z} + t_p(\nu)s(\bar{z})\right] = p$$

$\Rightarrow$ *The interval $\bar{z} \pm t_p(\nu)s(\bar{z})$ contains a fraction p of the distribution of the values that can be attributed to Z.*

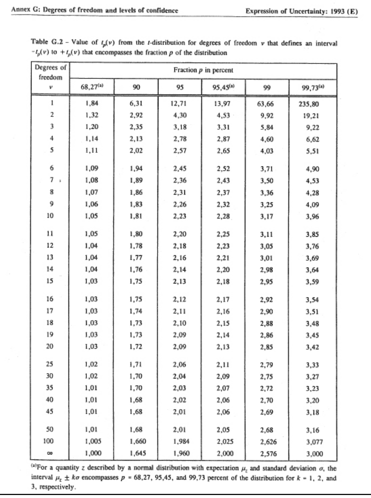


Table G.2 – Value of $t_p(\nu)$ from the t-distribution for degrees of freedom ν that defines an interval $-t_p(\nu)$ to $+t_p(\nu)$ that encompasses the fraction p of the distribution

Degrees of freedom ν	Fraction p in percent					
	68,27[(a)]	90	95	95,45[(a)]	99	99,73[(a)]
1	1,84	6,31	12,71	13,97	63,66	235,80
2	1,32	2,92	4,30	4,53	9,92	19,21
3	1,20	2,35	3,18	3,31	5,84	9,22
4	1,14	2,13	2,78	2,87	4,60	6,62
5	1,11	2,02	2,57	2,65	4,03	5,51
6	1,09	1,94	2,45	2,52	3,71	4,90
7	1,08	1,89	2,36	2,43	3,50	4,53
8	1,07	1,86	2,31	2,37	3,36	4,28
9	1,06	1,83	2,26	2,32	3,25	4,09
10	1,05	1,81	2,23	2,28	3,17	3,96
11	1,05	1,80	2,20	2,25	3,11	3,85
12	1,04	1,78	2,18	2,23	3,05	3,76
13	1,04	1,77	2,16	2,21	3,01	3,69
14	1,04	1,76	2,14	2,20	2,98	3,64
15	1,03	1,75	2,13	2,18	2,95	3,59
16	1,03	1,75	2,12	2,17	2,92	3,54
17	1,03	1,74	2,11	2,16	2,90	3,51
18	1,03	1,73	2,10	2,15	2,88	3,48
19	1,03	1,73	2,09	2,14	2,86	3,45
20	1,03	1,72	2,09	2,13	2,85	3,42
25	1,02	1,71	2,06	2,11	2,79	3,33
30	1,02	1,70	2,04	2,09	2,75	3,27
35	1,01	1,70	2,03	2,07	2,72	3,23
40	1,01	1,68	2,02	2,06	2,70	3,20
45	1,01	1,68	2,01	2,06	2,69	3,18
50	1,01	1,68	2,01	2,05	2,68	3,16
100	1,005	1,660	1,984	2,025	2,626	3,077
∞	1,000	1,645	1,960	2,000	2,576	3,000

[(a)] For a quantity z described by a normal distribution with expectation μ_z and standard deviation σ, the interval $\mu_z \pm k\sigma$ encompasses p = 68,27, 95,45, and 99,73 percent of the distribution for k = 1, 2, and 3, respectively.

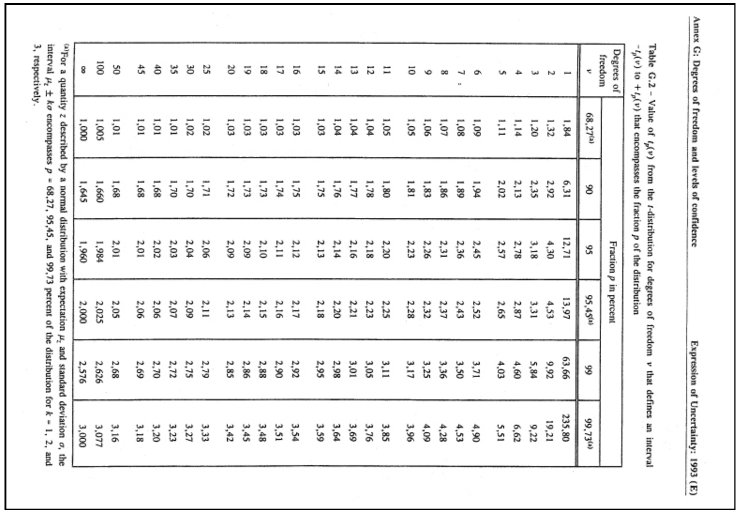


Table G.2 – Value of $t_p(\nu)$ from the t-distribution for degrees of freedom ν that defines an interval $-t_p(\nu)$ to $+t_p(\nu)$ that encompasses the fraction p of the distribution

Degrees of freedom ν	Fraction p in percent					
	68,27[a]	90	95	95,45[a]	99	99,73[a]
1	1,84	6,31	12,71	13,97	63,66	235,80
2	1,32	2,92	4,30	4,53	9,92	19,21
3	1,20	2,35	3,18	3,31	5,84	9,22
4	1,14	2,13	2,78	2,87	4,60	6,62
5	1,11	2,02	2,57	2,65	4,03	5,51
6	1,09	1,94	2,45	2,52	3,71	4,90
7	1,08	1,89	2,36	2,43	3,50	4,53
8	1,07	1,86	2,31	2,37	3,36	4,28
9	1,06	1,83	2,26	2,32	3,25	4,09
10	1,05	1,81	2,23	2,28	3,17	3,96
11	1,05	1,80	2,20	2,25	3,11	3,85
12	1,04	1,78	2,18	2,23	3,05	3,76
13	1,04	1,77	2,16	2,21	3,01	3,69
14	1,04	1,76	2,14	2,20	2,98	3,64
15	1,03	1,75	2,13	2,18	2,95	3,59
16	1,03	1,75	2,12	2,17	2,92	3,54
17	1,03	1,74	2,11	2,16	2,90	3,51
18	1,03	1,73	2,10	2,15	2,88	3,48
19	1,03	1,73	2,09	2,14	2,86	3,45
20	1,03	1,72	2,09	2,13	2,85	3,42
25	1,02	1,71	2,06	2,11	2,79	3,33
30	1,02	1,70	2,04	2,09	2,75	3,27
35	1,01	1,70	2,03	2,07	2,72	3,23
40	1,01	1,68	2,02	2,06	2,70	3,20
45	1,01	1,68	2,01	2,06	2,69	3,18
50	1,01	1,68	2,01	2,05	2,68	3,16
100	1,005	1,660	1,984	2,025	2,626	3,077
∞	1,000	1,645	1,960	2,000	2,576	3,000

[a] For a quantity z described by a normal distribution with expectation μ_z and standard deviation σ, the interval $\mu_z \pm k\sigma$ encompasses p = 68,27, 95,45, and 99,73 percent of the distribution for k = 1, 2, and 3, respectively.

Effective degrees of freedom

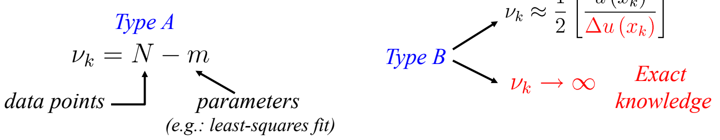
- Student's distribution holds only for normal variables rigorously.

$$Y = f\left(X_1, \ldots, X_N\right)$$

- Effective degrees of freedom (Welch-Satterthwaite formula)

$$\nu_{eff} = u_c^4(y) \Bigg/ \sum_{k=1}^{N} \left(\frac{\partial f}{\partial x_k}\right)^4 \frac{u^4\left(x_k\right)}{\nu_k}$$

- ν_k: degrees of freedom of the single quantity x_k



Type A

$$\nu_k = N - m$$

data points

parameters

(e.g.: least-squares fit)

Type B

$$\nu_k \approx \frac{1}{2}\left[\frac{u\left(x_k\right)}{\Delta u\left(x_k\right)}\right]^2$$

$$\nu_k \rightarrow \infty$$

Exact knowledge

Expanded uncertainty U (refined)

$$U_p = k_p \; u_c(y)$$

- Effective degrees of freedom: ν_{eff} .

$$Y = f\left(X_1, \ldots, X_N\right)$$

- t-factor for the desired level of confidence p: $k_p = t_p\left(\nu_{eff}\right)$.

- The previous results holds for approximately normal distribution of Y.

- If the combined uncertainty is dominated by a term evaluated from rectangular distribution the choice of $t_p(\nu_{eff}) > \sqrt{3}$ can lead to unacceptable results.

You can also find in the Guide ...

- More about correlated quantities, Student's distribution ...
- Theoretical basis of the ideas reported.
- How to report uncertainty (preferred formats, mandatory information).
- Practical suggestions.
- Examples.
- Short dictionary of general terms used in Metrology and basic statistical concepts.

Procedure for evaluating the uncertainty (I)

- Establish the relation between the measurand Y and ALL the input quantities X_k.

$$Y = f(X_1,\ldots,X_N)$$

f can be
→ mathematically known.
→ measurable.
→ a program.

- Evaluate estimates x_k of the input quantities X_k and the standard uncertainties $u(x_k)$ of such estimates (Type A and Type B evaluations).

Type A: set of independent observations available.

Type B: mathematically determined distributions, imported input values uncertainty of the method of measurement.

Procedure for evaluating the uncertainty (II)

- Compute y (the estimate of Y) using x_k.

$$y = f(x_1,\ldots,x_N)$$

f non-linear function:

$$y = \frac{1}{N}\sum_{k=1}^{N} f(X_{1,k},\ldots,X_{N,k})$$

- Compute the combined uncertainty.

$$u_c(y) = \sqrt{\sum_{k=1}^{N}\left(\frac{\partial f}{\partial x_k}\right)^2 u^2(x_k)}$$

- Compute the expanded uncertainty.

$$U_p = k_p\, u_c(y)$$

- Have I reported enough information in a sufficiently clear manner that my result can be updated in the future if new information or data become available?

Conclusions

- The uncertainty is an estimate of the likelihood of nearness to the best value that is consistent with the presently available knowledge.
- The method provides a realistic rather than a safe value of uncertainty.
- Evaluation of uncertainty components based on repeated observations (Type A) is not necessarily more reliable than that obtained by a "subjective" scientific judgement (Type B), especially when the number of observations is limited.

Bibliography

R.J. Barlow,
Statistics: a guide to the use of statistical methods in the physical sciences,
John Wiley & Sons, Chichester (England), 1995.

International vocabulary of basic and general terms in metrology,
International Organization for Standardization, Geneva, 1993.

International Organization for Standardization et al.,
Guide to the expression of uncertainty in measurement,
International Organization for Standardization, Geneva, 1993.