4.5 Cavity shape perturbation

Inserting small metallic objects into a cavity or slightly deforming the shape of the cavity can be treated by the perturbation technique [4], [5].

Let us designate the fields and the resonant frequency of the unperturbed cavity by E_0, H_0, ω_0 and of the perturbed cavity by E, H, ω; then Maxwell's equations are

$$\nabla \times \mathbf{E_0} = -j\,\omega_0\,\mu\,\mathbf{H_0}\,, \quad \nabla \times \mathbf{H_0} = j\,\omega_0\,\varepsilon\,\mathbf{E_0}\,, \tag{120}$$

$$\nabla \times \mathbf{E} = -j\,\omega\,\mu\,\mathbf{H}\,, \quad \nabla \times \mathbf{H} = j\,\omega\,\varepsilon\,\mathbf{E}\,, \tag{121}$$

Manipulating Eqs. (120) and (121) we find

$$\mathbf{H}\cdot(\nabla\times\mathbf{E_0^*}) - \mathbf{E_0^*}\cdot(\nabla\times\mathbf{H}) = \nabla\cdot(\mathbf{E_0^*}\times\mathbf{H}) = j\,\omega_0\,\mu\,\mathbf{H}\cdot\mathbf{H_0^*} - j\,\omega\,\varepsilon\,\mathbf{E_0^*}\cdot\mathbf{E}\,,$$
$$\mathbf{H_0^*}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{H_0^*}) = \nabla\cdot(\mathbf{E}\times\mathbf{H_0^*}) = -j\,\omega\,\mu\,\mathbf{H_0^*}\cdot\mathbf{H} + j\,\omega_0\,\varepsilon\,\mathbf{E}\cdot\mathbf{E_0^*}\,,$$

and after adding both equations and integration over the volume V of the perturbed cavity

$$\int_V \nabla\cdot(\mathbf{E}\times\mathbf{H_0^*} + \mathbf{E_0^*}\times\mathbf{H})\,dV = \oint_S (\mathbf{E}\times\mathbf{H_0^*} + \mathbf{E_0^*}\times\mathbf{H})\,d\mathbf{S}$$
$$= \oint_S (\mathbf{E_0^*}\times\mathbf{H})\,\cdot d\mathbf{S} = -j(\omega-\omega_0)\int_V (\varepsilon\,\mathbf{E}\cdot\mathbf{E_0^*} + \mu\,\mathbf{H}\cdot\,\mathbf{H_0^*})\,dV. \tag{122}$$

In deriving Eq. (122) we used Gauss' theorem and the fact that $n \times E = 0$ on the surface S of the perturbed cavity. Referring to Fig. 19, we see that $S = S_0 - \Delta S$ and write for the left side of Eq. (122)

$$\oint_S (\mathbf{E_0^*}\times\mathbf{H})\,d\mathbf{S} = \oint_{S_0} (\mathbf{E_0^*}\times\mathbf{H})\,d\mathbf{S} - \oint_{\Delta S} (\mathbf{E_0^*}\times\mathbf{H})\,d\mathbf{S} = -\int_{\Delta S} (\mathbf{E_0^*}\times\mathbf{H})\,d\mathbf{S}\,,$$

because $n \times E_0 = 0$ on S_0. Substitution into Eq. (122) gives

$$\omega - \omega_0 = -j\,\frac{\oint_{\Delta S}(\mathbf{E_0^*}\times\mathbf{H})\,d\mathbf{S}}{\int_V (\varepsilon\,\mathbf{E}\cdot\mathbf{E_0^*} + \mu\,\mathbf{H}\cdot\,\mathbf{H_0^*})\,dV}\,. \tag{123}$$

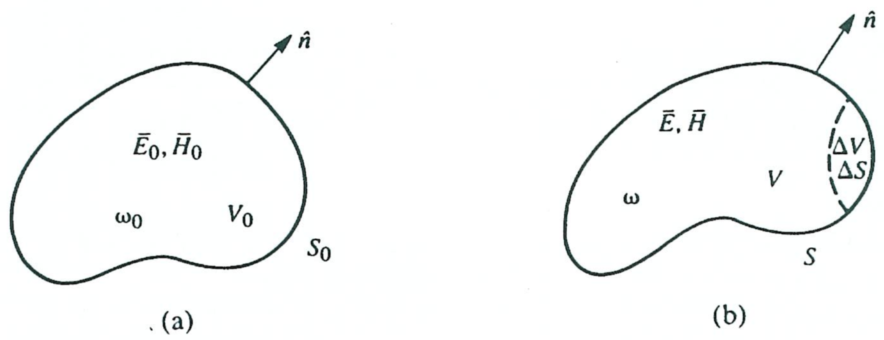

Fig. 19: Resonant cavity perturbed by a change in shape: (a) original cavity; (b) perturbed cavity

Equation (123) is the exact expression for the change in the resonant frequency. However, it is of little use since we do not know the quantities E, H of the perturbed cavity. But if the perturbation is small, E, H can be replaced by E_0, H_0 in the denominator of Eq. (123) because it is essentially the stored energy in the cavity and this will not change much. In the numerator we approximate H by H_0 and use Poynting's theorem,

$$\oint_{\Delta S} (\mathbf{E}_0^* \times \mathbf{H})\, d\mathbf{S} \approx \oint_{\Delta S} (\mathbf{E}_0^* \times \mathbf{H}_0)\, d\mathbf{S} = -j\,\omega_0 \int_{\Delta V} (\varepsilon\,|\mathbf{E}_0|^2 - \mu\,|\mathbf{H}_0|^2)\, dV\,,$$

which finally gives for Eq. (123)

$$\frac{\omega - \omega_0}{\omega_0} \approx \frac{\int_{\Delta V} (\mu\,|\mathbf{H}_0|^2 - \varepsilon\,|\mathbf{E}_0|^2)\, dV}{\int_{V_0} (\mu\,|\mathbf{H}_0|^2 + \varepsilon\,|\mathbf{E}_0|^2)\, dV} = \frac{\Delta W_m - \Delta W_e}{W_m + W_e}\,. \tag{124}$$

The terms ΔW_m, ΔW_e are the changes in the stored magnetic and electric energy, respectively, and $W_m + W_e$ is the total stored energy. The result shows that the frequency may either increase or decrease depending on the location and the character of the perturbation.

The formula (124) was derived by pushing the cavity wall inwards by a small amount. It seems reasonable to suppose that introducing a small metallic object into the interior of the cavity should perturb the frequency in a similar way by an amount depending upon the local fields, and thus we could use the frequency shift to measure the field strength at an interior point. This is in fact the case. We might further suppose that we only have to perform the integration of the unperturbed fields over the volume of the perturbing object. This, however, is far from the case because the object perturbs the field in a way that is essential. In order to calculate the field perturbation we follow a procedure for a small metallic sphere as outlined in Ref. [6]. With the well-known electric field of a metallic sphere in a homogeneous electrostatic field the volume integral over the electric field was performed when changing the sphere radius from r_0 to $r_0 + dr_0$. For the total perturbation caused by the sphere of radius r_0 the resulting expression was integrated from zero to r_0. In an analogous manner the volume integral over the magnetic field was carried out. As a result form factors for the volume integrals in the numerator of Eq. (124) were found:

$$f_e = 3/2, \quad f_m = 3/4\,.$$

In general, these form factors depend on the shape and orientation and material of the perturbing object. For some geometries, like ellipsoids, they are calculated [6]; for other more complicated geometries they can be determined experimentally [7].

5 MEASUREMENTS

Measurement techniques are a vast and complicated area. Here, I present a few basic techniques directly related to the subjects treated in the previous section.

5.1 Line mismatch

An old-fashioned but instructive way to measure a line mismatch is with a slotted line, Fig. 20. A movable capacitive probe measures the voltage standing wave ratio, Eq. (33), along the mismatched line. This yields the magnitude of the reflection coefficient. We further know from Section 2.3 that the first voltage minimum occurs at a distance $\zeta_{\min}$ from the load

$$2\beta\,\zeta_{\min} = \vartheta - \pi$$