REVIEW OF THEORY (I, II, III)

T. Weiland, M. Krasilnikov, R. Schuhmann, A. Skarlatos, M. Wilke
Darmstadt Technical University, Darmstadt, Germany

Abstract
After an introduction of Maxwell's equations in their most general form, an overview of the application of these equations to different types of field problems is given. The scalar and the vector wave equations are derived and solved for plane waves and guided waves in hollow waveguides. Some basic properties of modes in such guides and in cavities are discussed.

1 BASIC THEORY OF ELECTROMAGNETIC FIELDS

1.1 Maxwell's equations

The theory of electromagnetic fields is mainly involved with the solution of the set of *Maxwell's Equations* formulated by J.C. Maxwell in 1873 [1], which summarize all macroscopic phenomena of electricity. The solutions of these equations, adapted to various kinds of special problems, cover a wide range of applications. Following many other textbooks, Maxwell's equations are introduced axiomatically at the beginning of these notes. For the vector quantities

$$\vec{E}, \vec{H} \quad \text{(electric and magnetic field)}\ , \tag{1}$$

$$\vec{D}, \vec{B} \quad \text{(electric and magnetic flux density)}\ , \tag{2}$$

$$\vec{J} \quad \text{(electric current density)}\ , \tag{3}$$

and the scalar field

$$\rho \quad \text{(electric charge density)} \tag{4}$$

their integral form reads

$$\forall A, V \subset \Omega \ \text{ and } \ \forall t \in R:$$

$$\oint_{\partial A} \vec{E}(\vec{r},t) \cdot \mathrm{d}\vec{s} = -\frac{d}{dt} \int_A \vec{B}(\vec{r},t) \cdot \mathrm{d}\vec{A}\ , \tag{5}$$

$$\oint_{\partial A} \vec{H}(\vec{r},t) \cdot \mathrm{d}\vec{s} = \int_A \left(\frac{\partial \vec{D}(\vec{r},t)}{\partial t} + \vec{J}(\vec{r},t) \right) \cdot \mathrm{d}\vec{A}\ , \tag{6}$$

$$\oint_{\partial V} \vec{D}(\vec{r},t) \cdot \mathrm{d}\vec{A} = \int \rho(\vec{r},t)\ dV\ , \tag{7}$$

$$\oint_{\partial V} \vec{B}(\vec{r},t) \cdot \mathrm{d}\vec{A} = 0\ . \tag{8}$$

$A, V \subset \Omega$ are arbitrary faces or volumina in the underlying problem space $\Omega \subset R^3$, and $\partial A, \partial V$ are their boundaries. Equation (5) is called Faraday's law, Eq. (6) is Ampere's law.

By use of the theorems of Gauss and Stokes we can derive the differential form of Maxwell's equations:

$$\begin{aligned}
\operatorname{curl}\vec{E}(\vec{r},t) &= -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}\,, && (9)\\
\operatorname{curl}\vec{H}(\vec{r},t) &= \frac{\partial \vec{D}(\vec{r},t)}{\partial t} + \vec{J}(\vec{r},t)\,, && (10)\\
\operatorname{div}\vec{D}(\vec{r},t) &= \rho(\vec{r},t)\,, && (11)\\
\operatorname{div}\vec{B}(\vec{r},t) &= 0\,. && (12)
\end{aligned}$$

1.1.1 Some generalizations of Maxwell's equations

With a discrete set of point charges we have

$$\int \rho(\vec{r},t)dV = \sum_i Q_i(t) = Q(t)\,, \tag{13}$$

where the summation includes all point charges inside the integration volume. Together with Eq. (7) this leads to Gauss' law of electrostatics:

$$\oint_{\partial V} \vec{D}\cdot \mathrm{d}\vec{A} = Q\,. \tag{14}$$

In Eqs. (6) and (10)

$$\frac{\partial \vec{D}(\vec{r},t)}{\partial t} + \vec{J}(\vec{r},t) = \vec{J}_{tot}(\vec{r},t) \tag{15}$$

defines the total electric current density $\vec{J}_{tot}$, including the displacement current density $\partial\vec{D}/\partial t$ and the electric current density $\vec{J}$. The electric current density itself may consist of three parts: the conduction current density $\vec{J}_c$, the convection current density $\vec{J}_{cv}$, and the impressed current density $\vec{J}_i$:

$$\vec{J}(\vec{r},t) = \vec{J}_c(\vec{r},t) + \vec{J}_{cv}(\vec{r},t) + \vec{J}_i(\vec{r},t)\,. \tag{16}$$

The conduction current density $\vec{J}_c$ is only a function of the electric field $\vec{E}$. In isotropic and linear conductors with the conductivity κ it is given by Ohm's law:

$$\vec{J}_c(\vec{r},t) = \kappa\,\vec{E}(\vec{r},t)\,. \tag{17}$$

The convection current density $\vec{J}_{cv}$ covers free charges accelerated by the Lorentz force $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$:

$$\vec{J}_{cv}(\vec{r},t) = \rho(\vec{r},t)\,\vec{v}(\vec{r},t)\,. \tag{18}$$

The impressed current density $\vec{J}_i$ is assumed to be an externally enforced motion of charges, which is independent of all field forces (source term of electromagnetic fields).

The generalized form of Faraday's law is then

$$\oint_{\partial A} \vec{H}(\vec{r},t)\cdot \mathrm{d}\vec{s} = \int_A \left(\frac{\partial \vec{D}(\vec{r},t)}{\partial t} + \kappa\,\vec{E}(\vec{r},t) + \rho(\vec{r},t)\,\vec{v}(\vec{r},t) + \vec{J}_i(\vec{r},t)\right)\cdot \mathrm{d}\vec{A}\,. \tag{19}$$

In Ampere's law, Eq. (5), $\frac{d}{dt}$ denotes the total time derivative of the flux $\int \vec{B}\cdot \mathrm{d}\vec{A}$. This flux may change in time either by changing the magnetic flux density $\vec{B}$, or by changing the size or position of the face A. From vector analysis we get for faces moving with the velocity $\vec{v}$:

$$\oint_{\partial A} \vec{E}(\vec{r},t)\cdot \mathrm{d}\vec{s} = -\int_A \frac{\partial \vec{B}(\vec{r},t)}{\partial t}\cdot \mathrm{d}\vec{A} + \oint_{\partial A} (\vec{v}\times\vec{B}(\vec{r},t))\cdot \mathrm{d}\vec{s}\,. \tag{20}$$

1.1.2 Time-harmonic fields

In many technical applications sinusoidal field quantities occur that can be described as the real part of complex quantities:

$$f(t) = \mathrm{Re}\{\underline{f}_\omega \, e^{i\omega t}\} = |\underline{f}_\omega| \, \cos(\omega t + \varphi) \, , \tag{21}$$

where ω is the angular frequency, φ the phase angle, and $\underline{f}_\omega$ the complex amplitude. For vector quantities, e.g. the electric field vector, we can write

$$\vec{E}(\vec{r}, t) = \mathrm{Re}\{\underline{\vec{E}}_\omega(\vec{r}) \, e^{i\omega t}\} \, . \tag{22}$$

The vector complex amplitude $\underline{\vec{E}}_\omega(\vec{r})$ is also referred to as phasor. It is important to note that these complex amplitudes themselves are not physical quantities, but that the fields have to be derived by multiplying by $e^{i\omega t}$ and taking the real part.

The advantage of the complex notation is that all time derivatives can be replaced by simple multiplications with $i\omega$:

$$\frac{\partial \vec{E}(\vec{r}, t)}{\partial t} = \mathrm{Re}\{i\omega \, \underline{\vec{E}}_\omega(\vec{r}) \, e^{i\omega t}\} \, . \tag{23}$$

Maxwell's equations in stationary media can then be written as (indices ω are omitted):

$$\oint_{\partial A} \underline{\vec{E}}(\vec{r}) \cdot \mathrm{d}\vec{s} = -\int_A i\omega \underline{\vec{B}}(\vec{r}) \cdot \mathrm{d}\vec{A} \, , \tag{24}$$

$$\oint_{\partial A} \underline{\vec{H}}(\vec{r}) \cdot \mathrm{d}\vec{s} = \int_A (i\omega \underline{\vec{D}}(\vec{r}) + \underline{\vec{J}}(\vec{r})) \cdot \mathrm{d}\vec{A} \, , \tag{25}$$

$$\oint_{\partial V} \underline{\vec{D}}(\vec{r}) \cdot \mathrm{d}\vec{A} = 0 \, , \tag{26}$$

$$\oint_{\partial V} \underline{\vec{B}}(\vec{r}) \cdot \mathrm{d}\vec{A} = 0 \, . \tag{27}$$

Usually, the absence of free charges is assumed in this time-harmonic case (equivalent to fields in steady-state).

1.2 Fields inside matter

To complete the system of Maxwell's equations, the induction quantities $\vec{D}$ and $\vec{B}$ have to be linked to the field quantities $\vec{E}$ and $\vec{H}$. In their most general form, the constitutive equations are

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \, , \tag{28}$$

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} \, , \tag{29}$$

introducing the electric polarization $\vec{P}$ (also referred to as electric dipole moment density), and the magnetic polarization $\mu_0 \vec{M}$. The vector $\vec{M}$ is called magnetization. The constants

$$\varepsilon_0 \approx 8.854 \cdot 10^{-12} \frac{\mathrm{As}}{\mathrm{Vm}}, \quad \mu_0 = 4\pi \cdot 10^{-7} \frac{\mathrm{Vs}}{\mathrm{Am}} \tag{30}$$

are the permittivity and the permeability of vacuum, respectively.

The polarization quantities describe the influence of matter on electromagnetic fields. For linear materials without hysteresis and without permanent polarization they are proportional to the field vectors. The stationary relations are then given by

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \, , \tag{31}$$

$$\mu_0 \vec{M} = \mu_0 \chi_m \vec{H} \, , \tag{32}$$

with the electric and magnetic susceptibility χ_e, χ_m. For anisotropic materials the polarization vectors may have different orientations to the corresponding field vectors, which can be expressed by χ_e or χ_m becoming tensors.

1.2.1 Dielectric material characteristics

The properties of dielectric materials are based on the interaction of polarization charges (electrons or ions, in contrast to free charges in conductors) with external electromagnetic fields.

In the absence of external fields, atoms or molecules may or may not have electric dipole moments, but if they do, these are randomly oriented.[1] In the presence of a field, the atoms become polarized (or their permanent moments tend to align with the field). The dipole moment of a single polarization charge can be modelled by a pair of two point charges $\pm q$ and is expressed by

$$\vec{p} = q\vec{x} \,, \tag{33}$$

where $\vec{x}$ is the distance between the charges (see Fig. 1).

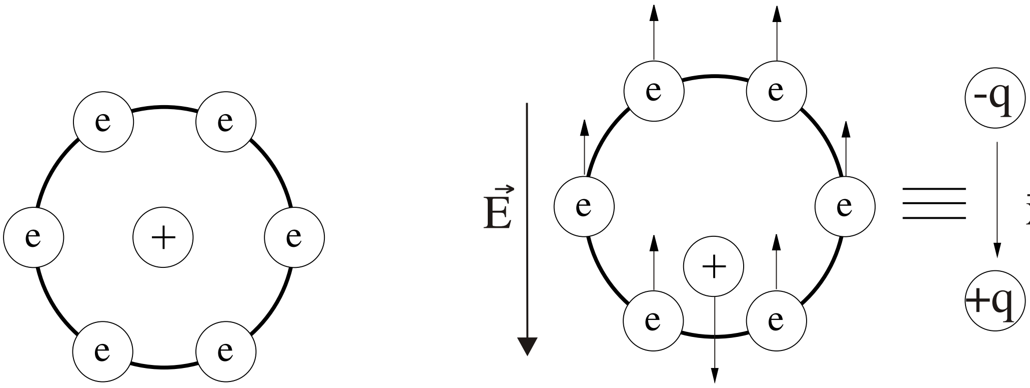

Fig. 1: Modelling of the displacement of atomic polarization charges by an electric dipole

For a set of N dipoles per unit volume with identical orientation, the macroscopic properties can be summarized in the polarization

$$\vec{P} = N\vec{p} = Nq\vec{x} \,. \tag{34}$$

Generally the polarization $\vec{P}$ is a non-linear function of the electric field $\vec{E}$, and can be expanded in terms of a power series. For many technical applications, it is sufficient to take only the linear terms into account. This leads to the susceptibility χ_e in Eq. (31), and for the flux density to

$$\begin{aligned} \vec{D} &= \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E} \\ &= \varepsilon_0 \varepsilon_r \vec{E} \end{aligned} \tag{35}$$

with the relative permittivity ε_r (isotropic, stationary case). This linearization is also very common, if small signals are superimposed on a steady value of $\vec{E}$.

For non-stationary fields the displacement of polarization charges is a dynamic process that can be modelled by several types of differential equations. In the Debye model the first order equation (for relaxation processes)

$$\frac{\partial}{\partial t} P(t) = \frac{1}{\tau} [(\varepsilon_s - \varepsilon_\infty) \varepsilon_0 \, E(t) - P(t)] \tag{36}$$

[1] Except for electrets, which have a permanent polarization.

leads to the complex permittivity in the frequency domain

$$\underline{\varepsilon}_r(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + i\omega\tau} \ . \tag{37}$$

Resonant effects (e.g. for electronic polarization) can be described by the Lorentz model using a second order differential equation. The frequency dependence on the relative permittivity as a result of a superposition of several polarization mechanisms is shown in Fig. 2.

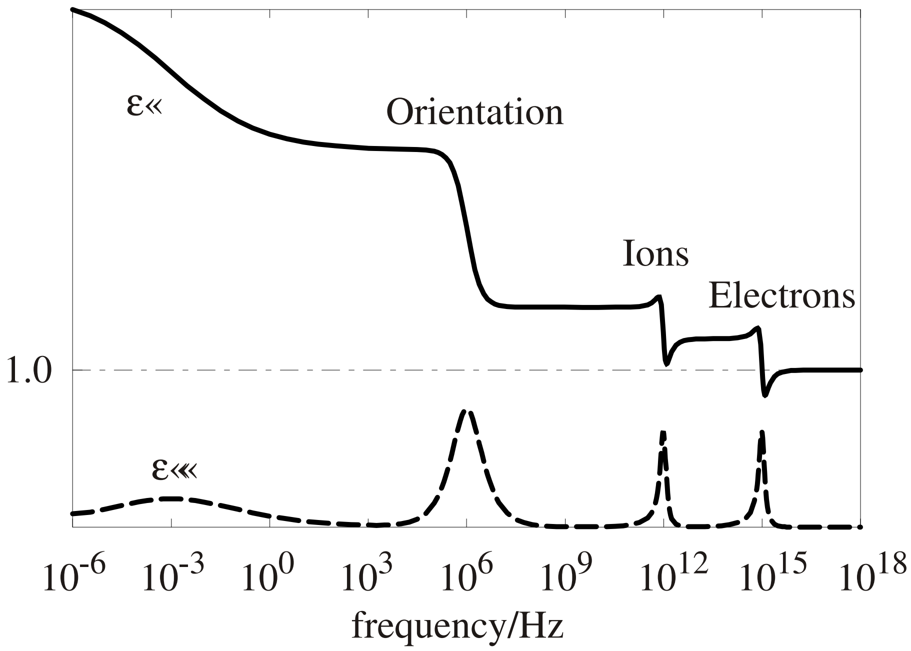

Fig. 2: Typical frequency dependence of the complex permittivity $\underline{\varepsilon}(\omega) = \varepsilon' - i\varepsilon''$ as a result of different types of polarization: orientation of permanent molecular dipoles (Debye model) and polarization of ions and electrons (Lorentz model).

A complex permittivity can also be used in conductors with a non-zero conductivity κ. Summarizing the current terms on the right-hand side of Faraday's law, Eq. (25), by

$$i\omega\underline{\vec{D}} + \underline{\vec{J}} = (i\omega\varepsilon + \kappa)\underline{\vec{E}} + \underline{\vec{J}}_i = i\omega(\varepsilon + \frac{\kappa}{i\omega})\underline{\vec{E}} + \underline{\vec{J}}_i \ , \tag{38}$$

the complex permittivity is defined as

$$\underline{\varepsilon} = \varepsilon + \frac{\kappa}{i\omega} \ . \tag{39}$$

Common notations for the complex permittivity are

$$\underline{\varepsilon} = \varepsilon' - i\varepsilon'' = \varepsilon' \left(1 - i\tan\delta_\varepsilon\right) \qquad (\tan\delta_\varepsilon = \varepsilon''/\varepsilon') \ , \tag{40}$$

where δ_ε is referred to as the electric loss angle of the medium.

In dielectrics we have $\tan\,\delta_\varepsilon \ll 1$ at most technical frequencies, and thus $\tan\,\delta_\varepsilon \approx \delta_\varepsilon$. On the contrary, in good conductors (metals) we have $\tan\,\delta_\varepsilon \gg 1$ and $\varepsilon \approx \varepsilon_0$, i.e. $\underline{\varepsilon}$ becomes almost purely imaginary.

1.2.2 Magnetic material characteristics

By analogy to electric polarization, the magnetic properties of materials can be described by means of a magnetic polarization vector, which is based on the orientation of atomic magnetic dipoles in the presence of external fields. Such a dipole can be modelled by a circulating atomic current i_c enclosing a circuit with the normal vector $\vec{A}$. The corresponding magnetic dipole moment is defined by

$$\vec{\mu} = i_c \vec{A} \, . \tag{41}$$

For N dipoles per unit volume with identical orientation, the (macroscopic) magnetization is given by

$$\vec{M} = N\vec{\mu} = N i_c \vec{A} \, . \tag{42}$$

Referring to the dependence of the polarization mechanism on the external magnetic field, magnetic materials are classified in diamagnetic, paramagnetic, and ferromagnetic substances. The polarization in diamagnetic and paramagnetic media is approximately proportional to the external field according to Eq. (32), and the relative permeability

$$\mu_r = 1 + \chi_m \tag{43}$$

generally differs only slightly from one (e.g. $\mu_r \approx 0.999$ for diamagnetic and $\mu_r \approx 1.001$ for paramagnetic media).

In ferromagnetic substances the relation between external field and magnetic polarization is non-linear and often depends on the preparation history of the material, leading to a hysteresis loop, as shown in Fig. 3.

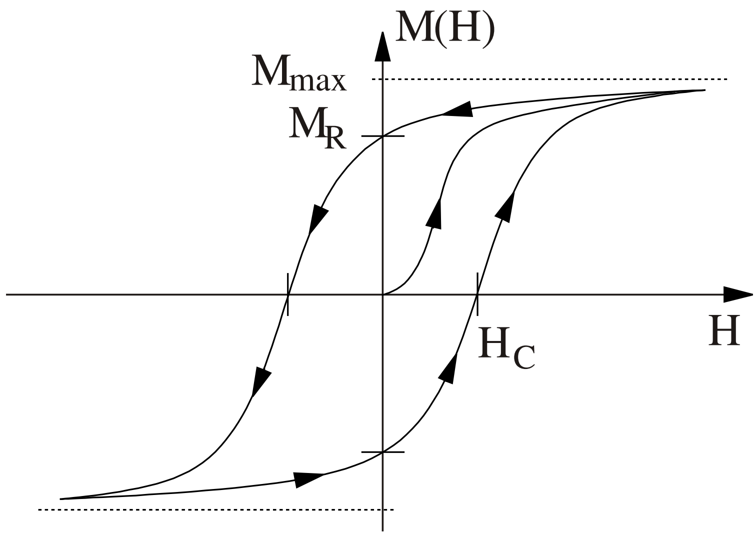

Fig. 3: Hysteresis loop of a ferromagnetic material including initial magnetization curve. H_C, M_R, and M_{max} are the coercive force, the remanence magnetization, and the saturation magnetization, respectively.

Referring to such curves, various kinds of permeability quantities can be defined. The normal permeability μ is given by the quotient B/H at the tip of the loop, and tables or graphs of μ as a function of H_{max} or B_{max} are frequently given in the literature. The differential permeability is defined as the derivative of B with respect to H, and can be as high as 10^6 for high-permeability materials. Most untreated ferromagnetic materials have a linear relation between $\vec{B}$ and $\vec{H}$ for small fields. Typical values of the initial permeability range from 10 to 10^4.

Like electric polarization, magnetization is also a dynamic process. Therefore the frequency dependence of the permeability has to be taken into account for RF fields. In the frequency domain this leads to the complex permittivity

$$\underline{\mu} = \mu' - i\mu'' = \mu' \left(1 - i \tan \delta_\mu\right) \tag{44}$$

with the magnetic loss angle δ_μ.

1.3 Applications of Maxwell's equations

The solutions of Maxwell's equations can be classified by several types of electromagnetic field, which can be handled by partly simplified equations: electrostatics, magnetostatics, stationary fields, quasi-stationary fields, harmonic fields, and general time-varying fields.

1.3.1 Static and stationary fields

The first simplification is based on the assumption that all field quantities are independent of time ($\partial/\partial t \equiv 0$). In this case we obtain the following system:

$$\oint_{\partial A} \vec{E}(\vec{r}) \cdot \mathrm{d}\vec{s} = 0 \,, \tag{45}$$

$$\oint_{\partial A} \vec{H}(\vec{r}) \cdot \mathrm{d}\vec{s} = \int_{A} \vec{J}(\vec{r}, t) \cdot \mathrm{d}\vec{A} \,, \tag{46}$$

$$\oint_{\partial V} \vec{D}(\vec{r}) \cdot \mathrm{d}\vec{A} = \int \rho(\vec{r})\, dV \,, \tag{47}$$

$$\oint_{\partial V} \vec{B}(\vec{r}) \cdot \mathrm{d}\vec{A} = 0 \,. \tag{48}$$

Obviously the electric-field vectors $\vec{E}$ and $\vec{D}$ and the magnetic vectors $\vec{H}$ and $\vec{B}$ do not depend on each other in current-free regions ($\vec{J} = 0$). In regions with current densities ($\vec{J} \neq 0$), they represent the sources of the magnetic fields, but there is no influence of the magnetic fields back on the electric quantities. Thus, we have two separated types of field problem: electrostatic fields and magnetic fields of stationary currents.

Electrostatic fields are governed by the equations (in differential form)

$$\mathrm{curl}\,\vec{E}(\vec{r}) = 0 \,, \tag{49}$$

$$\mathrm{div}\,\vec{D}(\vec{r}) = \rho(\vec{r}) \,. \tag{50}$$

A common approach is to fulfil Eq. (49) implicitly by defining the electric field as (negative) gradient of a scalar potential $\Phi(\vec{r})$:

$$\vec{E}(\vec{r}) = -\mathrm{grad}\,\Phi(\vec{r}) \quad \Rightarrow \quad \mathrm{curl}\,\vec{E}(\vec{r}) = -\mathrm{curl}\ \mathrm{grad}\ \Phi(\vec{r}) \equiv 0 \,. \tag{51}$$

This leads to the general equation for the scalar potential function

$$\mathrm{div}\ \varepsilon(\vec{r})\,\mathrm{grad}\ \Phi(\vec{r}) = -\rho(\vec{r}) \,. \tag{52}$$

In homogeneous regions (constant permittivity ε) we get Poisson's equation

$$\mathrm{div}\ \mathrm{grad}\,\Phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon} \,. \tag{53}$$

Finally, in regions where there is no charge density, the potential satisfies Laplace's equation

$$\mathrm{div}\,\mathrm{grad}\,\Phi(\vec{r}) = 0 \,. \tag{54}$$

For the various kinds of boundary value problems emerging from these types of equations, a wide range of specialized algorithms are available [2, 3]. As an example, Fig. 4 shows the potential as well as the electric field of a two-dimensional problem.

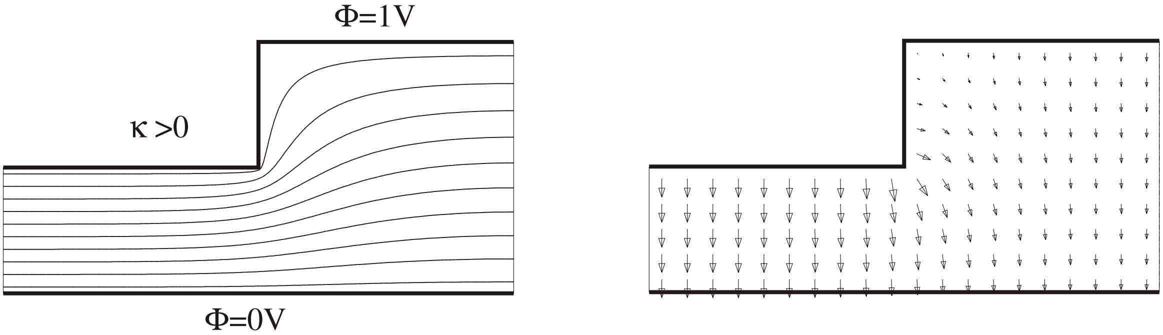

Fig. 4: Solution of an electrostatic boundary value problem: equipotential lines and electric field

An equation of the same type has to be solved for stationary current densities inside conductors. From

$$\begin{aligned} \mathrm{curl}\, \vec{E}(\vec{r}) &= 0, & (55) \\ \mathrm{div}\, (\kappa(\vec{r})\vec{E}(\vec{r})) = \mathrm{div}\, \vec{J}(\vec{r}) &= \mathrm{div}\, \mathrm{curl}\, \vec{H}(\vec{r}) = 0 & (56) \end{aligned}$$

we obtain the equation

$$\mathrm{div}\, \kappa(\vec{r})\, \mathrm{grad}\, \Phi(\vec{r}) = 0\ . \tag{57}$$

An example of a stationary current density field $\vec{J}(\vec{r}) = -\kappa(\vec{r})\, \mathrm{grad}\, \Phi(\vec{r})$ is shown in Fig. 5.

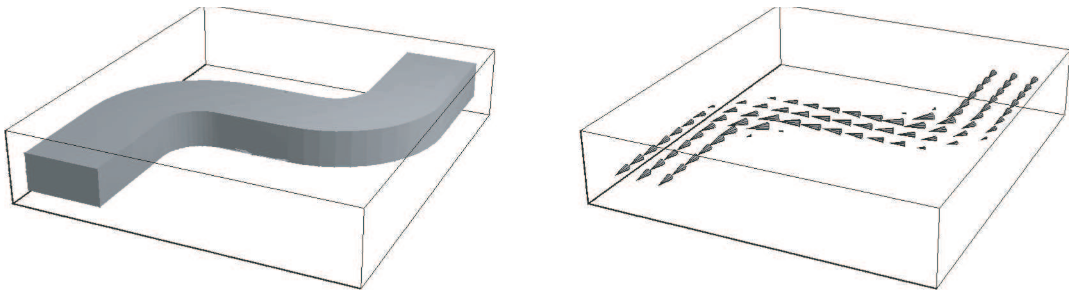

Fig. 5: Bent conductor and stationary current density

Conduction current densities as well as impressed currents are ultimately the sources for the magnetic fields of stationary currents (magnetostatic fields). Maxwell's equations now reduce to

$$\begin{aligned} \mathrm{curl}\, \vec{H}(\vec{r}) &= \vec{J}(\vec{r})\ , & (58) \\ \mathrm{div}\, \vec{B}(\vec{r}) &= 0\ . & (59) \end{aligned}$$

Because of the inhomogeneity on the right-hand side of Eq. (58), the approach with a scalar potential cannot be used here. Instead, a vector potential can be applied to fulfil Eq. (59) implicitly:

$$\vec{B}(\vec{r}) = \mathrm{curl}\, \vec{A}(\vec{r}) \quad \Rightarrow \quad \mathrm{div}\, \vec{B}(\vec{r}) = \mathrm{div}\ \mathrm{curl}\, \vec{A}(\vec{r}) \equiv 0\ . \tag{60}$$

Now the equation to solve for the potential function is

$$\mathrm{curl}\ \mu(\vec{r})^{-1}\mathrm{curl}\, \vec{A}(\vec{r}) = \vec{J}(\vec{r})\ . \tag{61}$$

For specialized approaches we again refer to the literature [2, 3]. As an example, Fig. 6 shows the distribution of the magnetic field $\vec{H}$ and the magnetic flux density $\vec{B}$ of a C-shaped magnet driven by two current coils.

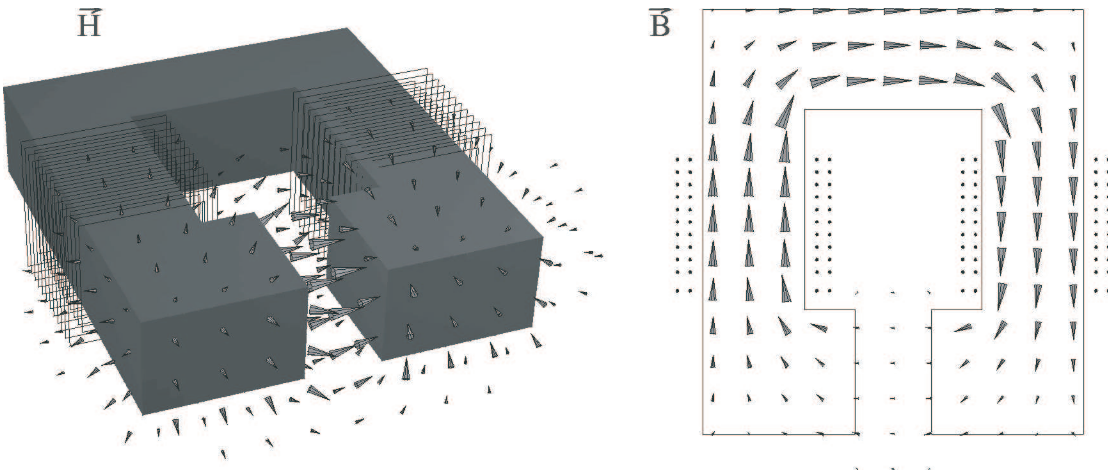

Fig. 6: Magnetic field $\vec{H}$ and magnetic flux density $\vec{B}$ of a C-shaped magnet driven by two current coils

1.3.2 *Quasi-stationary fields*

For quasi-stationary fields a certain time dependence is allowed, but all fields are assumed to be slowly varying. In this case, (one of) the time derivatives in Maxwell's equations can be approximated to zero.

As a first problem type, neglecting the displacement current in Eq. (10), we get magneto-quasistatic fields (see Fig. 7), with the equations

$$\operatorname{curl}\vec{E} = -\frac{\partial\vec{B}}{\partial t}\,, \tag{62}$$

$$\operatorname{curl}\vec{H} = \vec{J}_i + \kappa\vec{E}\,. \tag{63}$$

To validate the assumption $\partial\vec{D}/\partial t \approx 0$, an electric field $\vec{E}_0(\vec{r},t)$ is considered, which is switched on during the time period T. With $\varepsilon(\vec{r}) = \varepsilon_0$ the displacement current can be approximated by

$$\left|\frac{\partial\vec{D}(\vec{r},t)}{\partial t}\right| \approx \frac{\varepsilon_0|\vec{E}_0(\vec{r},t)|}{T}\,. \tag{64}$$

For the conduction current in a conducting material (conductivity κ) we have

$$|\vec{J}(\vec{r},t)| = \kappa|\vec{E}_0(\vec{r},t)|\,. \tag{65}$$

Therefore neglecting the displacement current is valid for

$$\frac{\varepsilon_0}{\kappa T} \ll 1\,, \tag{66}$$

which is usually the case for slowly varying fields in good conductors; e.g. in copper with $T = 1$ ms:

$$\frac{\varepsilon}{\kappa T} \approx \frac{8.85\cdot 10^{-12}As/Vm}{5.8\cdot 10^{7}(\Omega m)^{-1}\ 10^{-3}\,s} \approx 1.5\cdot 10^{-16} \ll 1\,.$$

In the non-conducting regions of the problem domain (e.g. the surrounding air) the quasi-static approximation $|\partial\vec{D}_{air}/\partial t| \ll |\vec{J}_{air}|$ never holds, as we have $\vec{J}_{air} = 0$. However, if we are only interested in the fields inside the conductor, a quasi-static approach is permissible if the maximum value of the displacement current is much smaller than the conducting currents:

$$\max_{\vec{r}\in\Omega}\left|\frac{\partial\vec{D}(\vec{r})}{\partial t}\right| \ll \max_{\vec{r}\in\Omega}|\vec{J}_\kappa(\vec{r})|\,. \tag{67}$$

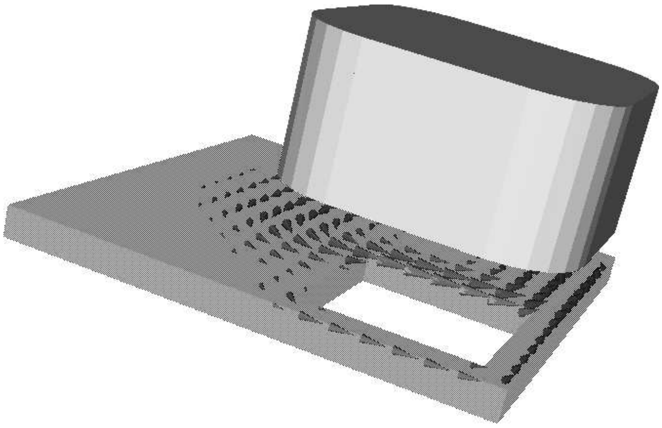

Fig. 7: Magneto-quasistatic fields: eddy currents in a copper plate with a hole induced by a coil excitation at 50 Hz

For some applications the time derivative of the magnetic flux density in Eq. (9) can also be neglected, leading to (in frequency domain)

$$\operatorname{curl}\vec{E} = 0\,, \tag{68}$$
$$\operatorname{curl}\vec{H} = i\omega\vec{D} + \kappa\vec{E}\,. \tag{69}$$

Similar to electrostatics, such electro-quasistatic fields can be expressed in terms of a complex-valued scalar potential $\underline{\Phi}$:

$$\underline{\vec{E}} = -\operatorname{grad}\underline{\Phi} \quad \Rightarrow \quad \operatorname{curl}\underline{\vec{E}} = -\operatorname{curl}\operatorname{grad}\underline{\Phi} \equiv 0 \tag{70}$$
$$\operatorname{div}\,(i\omega\varepsilon + \kappa)\operatorname{grad}\underline{\Phi} = 0\,. \tag{71}$$

An example, the electro-quasistatic field in an insulator contaminated by several water drops, is shown in Fig. 8.

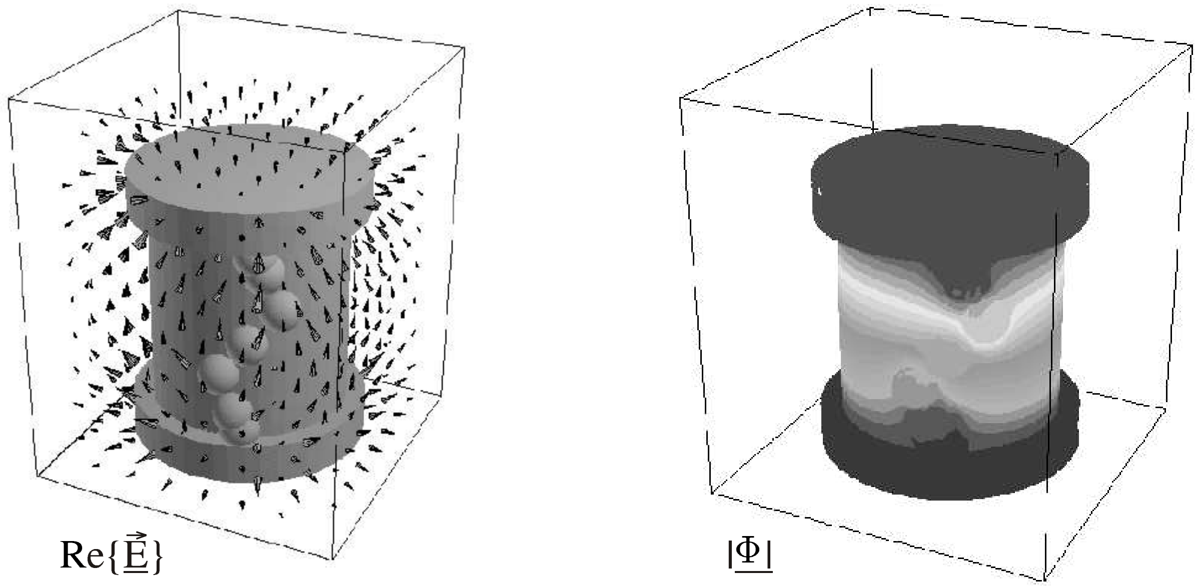

Fig. 8: High-voltage insulator contaminated by water drops at 50 Hz: real part of the electric phasor $\underline{\vec{E}}$ and absolute value of the complex potential $\underline{\Phi}$

1.3.3 Quickly varying fields

If the time derivatives of the fields cannot be neglected, the full set of Maxwell's equations have to be taken into account. In most analytical approaches — and also in the following parts of these notes — the time-harmonic formulation Eqs. (24)–(27) including the complex phasors is applied. However, especially in numerical field simulations, as well as in some measurement techniques, the usage of transient fields is also quite common.

1.4 Fields at interfaces between different media

Before we can solve electromagnetic boundary value problems, we must establish the boundary conditions satisfied by the field and flux vectors at the interface between two media with different material properties.

1.4.1 Boundary conditions for electric fields

To derive the boundary condition for the electric field $\vec{E}$ we consider the interface region shown in Fig. 9 and evaluate Faraday's law, Eq. (5), for the small circuit C_1. Assuming a very small size of C_1 we can

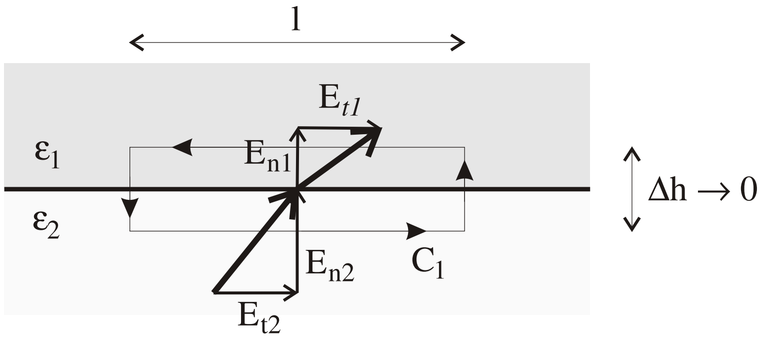

Fig. 9: Derivation of the continuity condition of the electric field. The vector fields $\vec{E}_1$ and $\vec{E}_2$ on either side of the interface are decomposed in their tangential and normal components.

approximate the fields in each of the media to be constant. We decompose them into their tangential (index 't') and their normal (index 'n') components and get

$$(E_{t2} - E_{t1}) \cdot l = \dot{B}_\perp \cdot l \, \Delta h \; , \tag{72}$$

where $B_\perp$ denotes the magnetic flux component normal to the circuit. If Δh tends to zero (keeping l fixed), the right-hand side vanishes, as the time derivative of the magnetic flux density has to be finite. This leads to the continuity of the tangential electric fields

$$E_{t2} = E_{t1} \qquad \text{or} \qquad \vec{n} \times (\vec{E}_1 - \vec{E}_2) = 0 \; , \tag{73}$$

where $\vec{n}$ is the normal vector of the interface plane.

For the normal components we consider the configuration in Fig. 10, where again the top and bottom surfaces of the cylinder are so small that constant fields can be assumed. Evaluating the surface integral on the left-hand side of Gauss' law, Eq. (7), we get

$$\oint \vec{D} \cdot \mathrm{d}\vec{A} \quad \xrightarrow{\Delta h \to 0} \quad (D_{n1} - D_{n2}) \cdot \Delta A \; , \tag{74}$$

where D_{n1} and D_{n2} are the normal components of the electric flux density on both sides of the interface. The right-hand side — the integral of the charge density inside the cylinder — vanishes for $\Delta h \to 0$, unless there is a non-zero surface charge density ρ_s at the interface plane. Therefore the boundary condition for the normal electric components reads as

$$D_{n1} - D_{n2} = \rho_s \qquad \text{or} \qquad \vec{n} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \; . \tag{75}$$

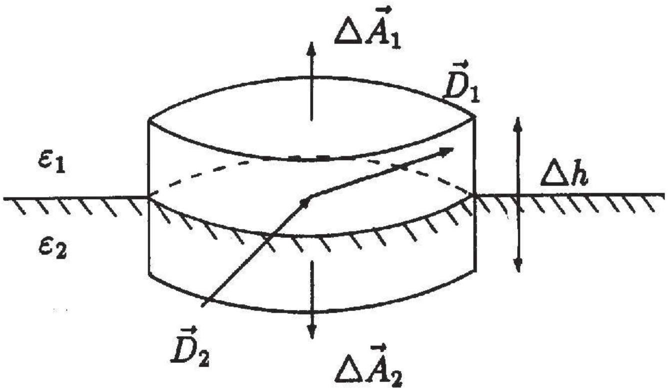

Fig. 10: Derivation of the continuity condition of the electric flux density

1.4.2 Boundary conditions for magnetic fields

Analogous configurations can be considered. For the magnetic fields, as there are no magnetic charges, the evaluation of the surface integral of the magnetic flux density (similar to Fig. 10) leads to the continuity of the normal magnetic flux density

$$\vec{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0 \,. \tag{76}$$

If we evaluate Faraday's law for the circuit shown in Fig. 11, the right-hand side is given as the total current through the face ΔA:

$$\int\limits_{\Delta A} (\vec{J} + \frac{\partial \vec{D}}{\partial t}) \cdot \mathrm{d}\vec{A} \,. \tag{77}$$

As the time derivative of the electric flux has to be finite, the integral tends to zero for $\Delta h \to 0$, unless there is an electric surface current J_s along the interface. Therefore the boundary condition for the tangential magnetic components reads as

$$H_{t2} - H_{t1} = J_s \qquad \text{or} \qquad \vec{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s \,. \tag{78}$$

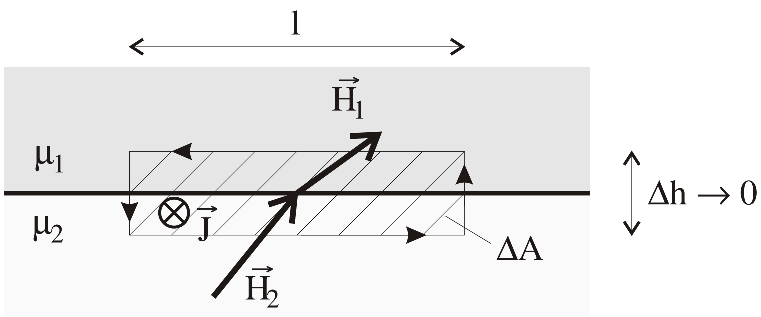

Fig. 11: Derivation of the continuity condition of the magnetic field

It can easily be shown that for non-stationary fields ($d/dt \neq 0$) it is always sufficient to impose the continuity of the tangential components of $\vec{E}$ and $\vec{H}$. This implicitly entails the continuity of the normal magnetic flux density; eventually the discontinuity of the normal components of $\vec{D}$ will yield the free electric surface charges on the interface.

1.4.3 *Infinitely conducting walls*

For many practical calculations the idealized model of infinitely conducting materials (also referred to as Perfect Electric Conductors, PEC) plays an important role. From $\vec{J} = \kappa\vec{E}$ we have $\vec{E} = 0$ inside the conductor, and from Ampere's law this implies $\partial\vec{B}/\partial t = 0$ and $\vec{B} = 0$ for non-stationary fields. The boundary conditions for the fields at the outer surface of the conductor then read as

$$\begin{aligned} \vec{n} \times \vec{E} &= 0 \,, & (79)\\ \vec{n} \times \vec{H} &= \vec{J}_s \,, & (80)\\ \vec{n} \cdot \vec{D} &= \rho_s \,, & (81)\\ \vec{n} \cdot \vec{B} &= 0 \,. & (82) \end{aligned}$$

1.5 Electric and magnetic energy

1.5.1 *Electric energy density*

The energy stored in an electromagnetic field must be identical to the energy that was required for establishing this field. To derive an expression for the energy of an electrostatic field, we consider the plate capacitor (surface ΔA, width Δs) shown in Fig. 12. The field vectors $\vec{E}$ and $\vec{D}$ inside are assumed to be homogeneous. The energy required to move a charge dQ (in the opposite direction to the electric

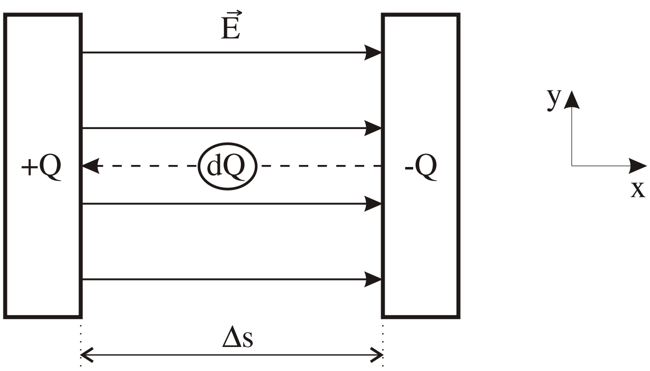

Fig. 12: Derivation of the electric energy density in a plate capacitor with surface ΔA and width Δs

field lines) from the right plate to the left amounts to

$$dW = d\vec{F} \cdot \Delta s\, \vec{e}_x = \vec{E}\, dQ \cdot \Delta s\, \vec{e}_x \,, \tag{83}$$

while the charge on the left plate increases by

$$dQ = d\vec{D} \cdot \Delta\vec{A} = d\vec{D} \cdot \Delta A\, \vec{e}_x \,. \tag{84}$$

The total energy of the field inside the capacitor volume $\Delta V = \Delta A\, \Delta s$ is given by

$$\Delta W = \int dW = \int \vec{E}(d\vec{D} \cdot \Delta A\, \vec{e}_x) \cdot \Delta s\, \vec{e}_x = \Delta V \int \vec{E} \cdot d\vec{D} \,. \tag{85}$$

The integral term is the stored energy per volume, or the energy density

$$w_e = \int \vec{E} \cdot d\vec{D} \,. \tag{86}$$

In linear isotropic media the energy density is given by

$$w_e = \frac{1}{2}\varepsilon|\vec{E}|^2 \,. \tag{87}$$

1.5.2 Magnetic energy density

A similar derivation can be performed for the energy in magnetic fields. We get the magnetic energy density

$$w_m = \int \vec{H} \cdot d\vec{B} \; , \tag{88}$$

and for linear isotropic media the simplified expression

$$w_m = \frac{1}{2}\mu|\vec{H}|^2 \; . \tag{89}$$

1.5.3 Energy densities of time-harmonic fields

The energy density of a time-harmonic electric field

$$\vec{E}(t) = \mathrm{Re}\{\underline{\vec{E}}e^{i\omega t}\} = \vec{E}_0 \, \cos(\omega t + \varphi) \tag{90}$$

in a linear isotropic medium is given by

$$\begin{aligned} w_e(t) &= \frac{1}{2}\varepsilon|\vec{E}(t)|^2 = \frac{1}{2}\varepsilon|\vec{E}_0|^2\cos^2(\omega t + \varphi) \\ &= \frac{1}{4}\varepsilon|\vec{E}_0|^2 \; (1 + \cos(2(\omega t + \varphi))) \; . \end{aligned} \tag{91}$$

In most cases we are interested in the time average of the energy during a period $T = 2\pi/\omega$, which amounts to

$$\overline{W}_e = \int\limits_V \overline{w}_e \mathrm{d}V \tag{92}$$

with

$$\begin{aligned} \overline{w}_e &= \frac{1}{T}\int\limits_{t_0}^{t_0+T} w_e(t)dt \\ &= \frac{1}{4}\varepsilon|\underline{\vec{E}}|^2 = \frac{1}{4}\varepsilon \; \underline{\vec{E}} \cdot \underline{\vec{E}}^* \; , \end{aligned} \tag{93}$$

where $\underline{\vec{E}}^*$ is the conjugate vector of $\underline{\vec{E}}$. The analogous expression for the time-averaged magnetic energy density is

$$\overline{w}_m = \frac{1}{4}\mu|\underline{\vec{H}}|^2 = \frac{1}{4}\mu \; \underline{\vec{H}} \cdot \underline{\vec{H}}^* \; . \tag{94}$$

2 ELECTROMAGNETIC WAVES

2.1 Plane-wave equation

2.1.1 The wave equation in time domain

According to Maxwell's equations, a time-varying electric field is coupled (by Ampere's law) to a magnetic field, the time variation of which is in turn coupled (by Faraday's law) to an electric field. Electromagnetic field patterns that propagate through space with a finite velocity are called electromagnetic waves.

The simplest form of an electromagnetic wave is the plane wave, propagating through free space (vacuum) or homogeneous media. For this type of wave a scalar wave equation can be derived from Maxwell's equations. At first we impose a number of restrictions:

- The medium contained in the solution domain is homogeneous, isotropic and linear.
- The medium is not conducting; $\kappa = 0$.
- There is no space charge distribution and no impressed-current density; $\rho = 0$, $\vec{J_i} = 0$.
- The fields vary only in one direction, say the x-axis; $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$.

Under these assumptions Maxwell's equations can be written as

$$\begin{aligned}
\text{curl}\,\vec{E}(\vec{r},t) &= -\mu\frac{\partial \vec{H}(\vec{r},t)}{\partial t}\,, && (95)\\
\text{curl}\,\vec{H}(\vec{r},t) &= \varepsilon\frac{\partial \vec{E}(\vec{r},t)}{\partial t}\,, && (96)\\
\text{div}\,\vec{E}(\vec{r},t) &= \text{div}\,\vec{H}(\vec{r},t) = 0\,. && (97)
\end{aligned}$$

As we supposed $\partial/\partial y = \partial/\partial z = 0$, the curl operator can be evaluated as

$$\begin{aligned}
\left(\text{curl}\,\vec{E}(x,t)\right)_y &= -\frac{\partial E_z(x,t)}{\partial x} = -\mu\frac{\partial H_y(x,t)}{\partial t}\,, && (98)\\
\left(\text{curl}\,\vec{E}(x,t)\right)_z &= \frac{\partial E_y(x,t)}{\partial x} = -\mu\frac{\partial H_z(x,t)}{\partial t}\,, && (99)\\
\left(\text{curl}\,\vec{H}(x,t)\right)_y &= -\frac{\partial H_z(x,t)}{\partial x} = \varepsilon\frac{\partial E_y(x,t)}{\partial t}\,, && (100)\\
\left(\text{curl}\,\vec{H}(x,t)\right)_z &= \frac{\partial H_y(x,t)}{\partial x} = \varepsilon\frac{\partial E_z(x,t)}{\partial t}\,. && (101)
\end{aligned}$$

Obviously we get two pairs of decoupled equations: Eqs. (98) and (101) with components E_z and H_y, and Eqs. (99) and (100) with components E_y and H_z. Eliminating the terms H_z and E_y from these equations we get

$$\left[\frac{\partial^2}{\partial x^2} - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right]\left\{\begin{array}{c} E_y(x,t)\\ H_z(x,t)\end{array}\right\} = 0\,. \qquad (102)$$

Both H_z and E_y satisfy the following differential equation

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right] f(x,t) = 0\,, \qquad (103)$$

where

$$v = \frac{1}{\sqrt{\mu\varepsilon}}\,. \qquad (104)$$

Equation (103) is called the homogeneous wave equation. For the solution of this equation in the time domain we use Alembert's transformation:

$$\begin{aligned}
\xi &= x - v\,t\,, && (105)\\
\eta &= x + v\,t\,, && (106)
\end{aligned}$$

leading to

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f(\xi,\eta)}{\partial \xi} + \frac{\partial f(\xi,\eta)}{\partial \eta}\,, & \frac{\partial^2 f}{\partial x^2} &= \frac{\partial^2 f(\xi,\eta)}{\partial \xi^2} + 2\,\frac{\partial^2 f(\xi,\eta)}{\partial \xi\,\partial \eta} + \frac{\partial^2 f(\xi,\eta)}{\partial \eta^2}\,,\\
\frac{\partial f}{\partial t} &= -v\frac{\partial f(\xi,\eta)}{\partial \xi} + v\frac{\partial f(\xi,\eta)}{\partial \eta}\,, & \frac{\partial^2 f}{\partial t^2} &= v^2\left[\frac{\partial^2 f(\xi,\eta)}{\partial \xi^2} - 2\,\frac{\partial^2 f(\xi,\eta)}{\partial \xi\,\partial \eta} + \frac{\partial^2 f(\xi,\eta)}{\partial \eta^2}\right]\,.
\end{aligned}$$

Substituting these expressions in Eq. (103) we get the simplified differential equation

$$\frac{\partial^2 f(\xi,\eta)}{\partial\xi\,\partial\eta} = 0 \,. \tag{107}$$

Its general solution is

$$\begin{aligned} f(\xi,\eta) &= F(\xi) + G(\eta)\,, & (108)\\ \Rightarrow f(x,t) &= F(x - v\,t) + G(x + v\,t)\,, & (109) \end{aligned}$$

with arbitrary functions F and G. As we can see in Eq. (109), the value of F at time instant $t = t_0$ on the plane $x = x_0$ is the same as the value at time $t = t_0 + \Delta t$ on the plane $x = x_0 + v\Delta t$. In other words the field distribution is moving towards the positive x-axis as the time t is running, and v is the velocity of the wave. In an analogous manner it can be shown that the field distribution expressed by G is moving towards the negative x-direction (see Fig. 13).

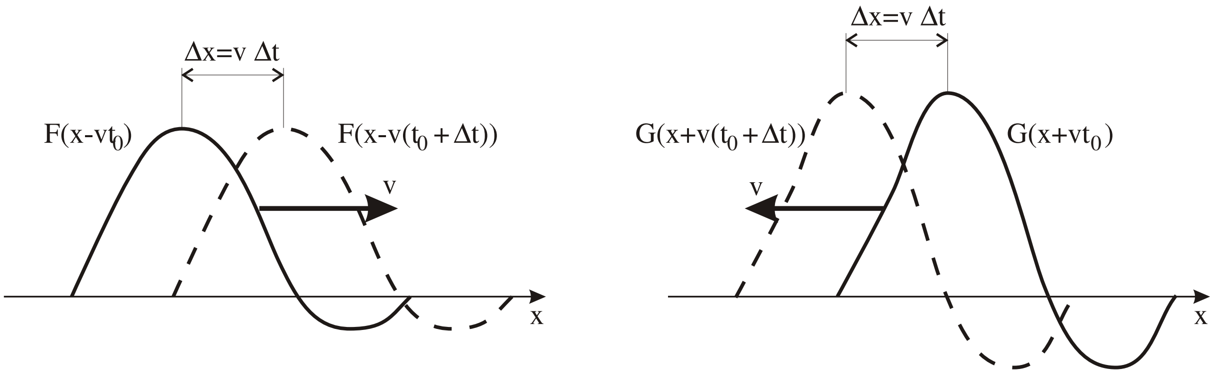

Fig. 13: Illustration of the functions F and G: spatial plots at different instants of time

Let us consider the solution for the electric field. From Eqs. (99) and (100) we get a solution for the E_y component:

$$E_y(x,t) = F(x - v\,t) + G(x + v\,t)\,. \tag{110}$$

The corresponding magnetic field can be found by evaluating Eq. (100):

$$\begin{aligned} \frac{\partial H_z}{\partial x} &= -\varepsilon\frac{\partial E_y}{\partial t} = \varepsilon\,v\,\left[F'(x - v\,t) - G'(x + v\,t)\right]\,, & \\ \Rightarrow H_z &= \varepsilon\,v\,\left[F(x - v\,t) - G(x + v\,t)\right]\,. & (111) \end{aligned}$$

A second, independent solution can be obtained from Eqs. (98) and (101), having field components E_z and H_y. Both solutions are referred to as the two linear polarizations of the plane wave. In general, a superposition of both solutions results in an elliptic polarization of the wave.

In Eqs. (110) and (111), the ratio

$$Z = \frac{E_y}{H_z} = \frac{1}{\varepsilon\,v} = \sqrt{\frac{\mu}{\varepsilon}} \tag{112}$$

is referred to as the wave impedance (or intrinsic impedance) of the medium. In vacuum we have

$$Z = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\,\Omega\,. \tag{113}$$

The solutions (110) and (111) represent an electromagnetic wave, whose components are normal to the direction of propagation. This kind of wave belongs to the category of plane waves, as the surfaces of equal phase and equal amplitude form planes.

In the beginning of the analysis we supposed that $\partial/\partial y = \partial/\partial z = 0$. This assumption implies that both the amplitude and the phase of the fields remain constant on the yz-plane. Thus the yz-planes are surfaces of both equal phase and equal amplitude. Waves of this kind are called homogeneous waves.

Putting it all together, the solution of the wave equation as derived in this paragraph is a homogeneous plane wave, see Fig. 14. This wave has no longitudinal field components, i.e. neither electric nor magnetic components in the direction of propagation (in our case $E_x = H_x = 0$). Therefore it belongs to the category of Transverse Electromagnetic (TEM) waves.

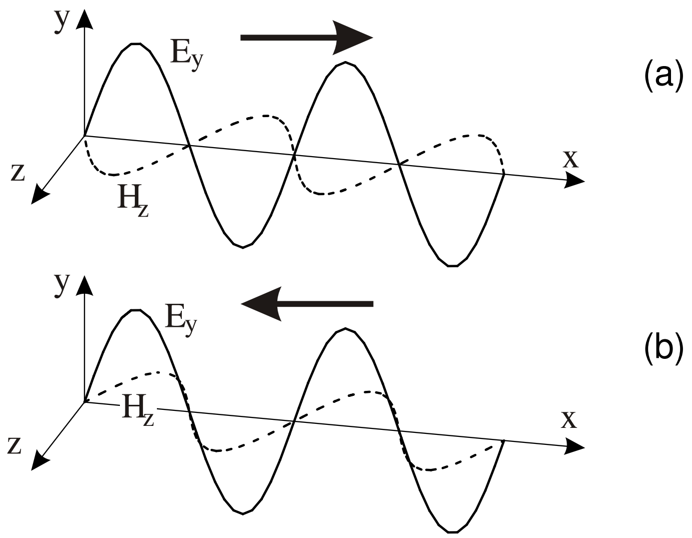

Fig. 14: Field components E_y and H_z of a homogeneous plane wave propagating in the (a) $+x$- or (b) $-x$-direction

In our analysis we assumed a wave propagation in the direction of the x-axes, which of course is not the general case. If the direction of propagation is defined by a unit vector $\vec{n}$, see Fig. 15, the solution of the wave equation is

$$f(\vec{r}, t) = f(\vec{n} \cdot \vec{r} - vt) \ . \tag{114}$$

The relation between the electric and magnetic field vectors is then given by

$$\vec{H} = \frac{1}{Z}\left(\vec{n} \times \vec{E}\right) \ . \tag{115}$$

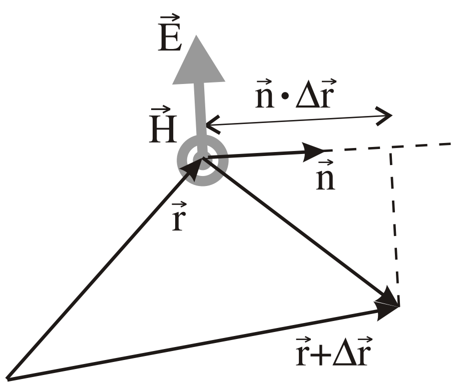

Fig. 15: Propagation of a plane wave in an arbitrary direction $\vec{n}$

2.1.2 Time-harmonic plane waves

In the previous section we obtained a solution of the wave equation inside an isotropic, homogeneous material for an arbitrary time variation. In many cases we are interested in problems in which the time variation is sinusoidal. In this case the complex notation for the time varying quantities can be used.

The assumptions we make here are the same as before, with the only difference being that the restriction of the non-conducting medium ($\kappa = 0$) is skipped. The rest of the assumptions remain the same: $\partial/\partial y = \partial/\partial z = 0$, ε and μ shall be constant, and no space-charge distribution is taken into account.

We define the complex propagation constant

$$\underline{\gamma} = \alpha + i\,\beta = i\underline{k} = i\omega\sqrt{\mu\underline{\varepsilon}}\,, \tag{116}$$

where the complex permittivity $\underline{\varepsilon}$ may or may not contain a conductivity term according to Eq. (39). The homogeneous wave equation (103) from time domain can then be transformed into the complex, homogeneous wave equation (without time variables)

$$\left[\frac{\partial^2}{\partial x^2} - \underline{\gamma}^2\right] \underline{f}(x) = 0\,. \tag{117}$$

A differential equation of this form is called a Helmholtz equation, and its solution is (for the electric field component $\underline{f} = \underline{E}_y$)

$$\underline{E}_y(x) = \underline{C}_1\, e^{-\underline{\gamma}\, x} + \underline{C}_2\, e^{+\underline{\gamma}\, x}\,, \tag{118}$$

The related magnetic field component $\underline{H}_z$ evaluates to

$$\underline{H}_z(x) = \frac{1}{\underline{Z}}\left[\underline{C}_1\, e^{-\underline{\gamma}\, x} - \underline{C}_2\, e^{+\underline{\gamma}\, x}\right]\,. \tag{119}$$

The term $\underline{Z}$ is the complex wave impedance:

$$\underline{Z} = \sqrt{\frac{\mu}{\underline{\varepsilon}}}\,. \tag{120}$$

Equations (118) and (119) describe the superposition of two independent waves propagating in the two opposite directions of the x-axis. The constant $\underline{\gamma} = \alpha + i\beta$ in Eq. (116) is called the propagation constant. From

$$\mathrm{Re}\{e^{-\underline{\gamma} x}\} = \mathrm{Re}\{e^{-\alpha x} \cdot e^{-i\beta x}\} = e^{-\alpha x} \cdot \cos(\beta x) \tag{121}$$

we find that the real part of $\underline{\gamma}$, the attenuation constant α, denotes the attenuation of the field amplitude, and the imaginary part, the phase constant β, denotes the phase shift along the direction of wave propagation (see Fig. 16). In a perfect dielectric (non-conducting) medium we get

$$\alpha = 0 \qquad \beta = \omega\sqrt{\varepsilon\mu}\,. \tag{122}$$

The time dependence of the field components in Eqs. (118) and (119) is implied in the complex notation. For example, the electric field propagating in the positive x-direction is given by

$$\begin{aligned} E_y(x,t) &= \mathrm{Re}\{\underline{C}_1\, e^{-\underline{\gamma}\, x}\, e^{i\omega\, t}\} = \mathrm{Re}\{|\underline{C}_1|\, e^{i\varphi}\, e^{-\alpha\, x - i\beta\, x + i\omega\, t}\} \\ &= |\underline{C}_1|\, e^{-\alpha\, x}\, \cos(\beta\, x - \omega\, t - \varphi) \\ &= |\underline{C}_1|\, e^{-\alpha\, x}\, \cos(\beta[x - v_p\, t] - \varphi) \qquad \text{with } v_p = \frac{\omega}{\beta}\,. \end{aligned} \tag{123}$$

The wavelength λ of a wave is defined as the spatial distance between two neighbouring planes of equal phase (see Fig. 17). From the relation $\beta\,\lambda = 2\,\pi$ we get

$$\lambda = \frac{2\,\pi}{\beta}\,. \tag{124}$$

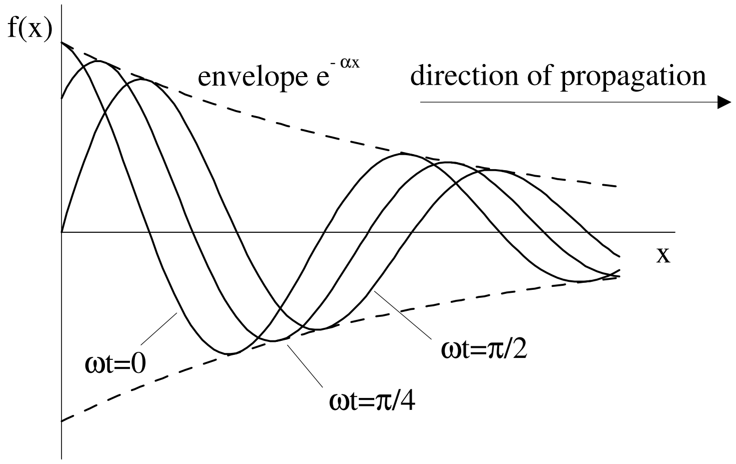

Fig. 16: Attenuated wave at several instants of time and envelope.

The period T of the wave is defined as

$$T = \frac{1}{f} = \frac{2\,\pi}{\omega} = \frac{2\,\pi}{\beta\,v_p}\,. \tag{125}$$

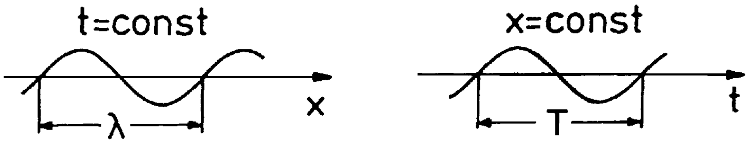

Fig. 17: Wavelength λ and period T of a harmonic plane wave

2.2 Waves in lossy matter, skin depth

In the general case of a lossy medium the propagation constant is a complex quantity, and a plane wave is attenuated in the direction of propagation according to $\exp(-\alpha x)$. As the wave impedance $\underline{Z}$ in Eq. (119) is complex too, $\underline{\vec{H}}$ is no longer in phase with $\underline{\vec{E}}$ (usually $\underline{\vec{H}}$ lags behind $\underline{\vec{E}}$). A sketch of $\underline{\vec{E}}$ and $\underline{\vec{H}}$ versus x at some instant of time would be similar to Fig. 16.

Two cases of particular interest are (i) good dielectrics (low-loss), and (ii) good conductors (high-loss). Both cases can be modelled by a complex permittivity quantity $\underline{\varepsilon} = \varepsilon' - i\varepsilon''$, where the imaginary part ε'' expresses the losses of the medium.

In the low-loss case, we have $\varepsilon'' \ll \varepsilon'$ and

$$\sqrt{\varepsilon' - i\varepsilon''} \approx \sqrt{\varepsilon'} - i\frac{\varepsilon''}{2\sqrt{\varepsilon'}} \qquad \Rightarrow \qquad \alpha \approx \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} \qquad\qquad \beta \approx \omega\sqrt{\mu\varepsilon'} \tag{126}$$

$$|\underline{Z}| \approx \sqrt{\frac{\mu}{\varepsilon'}} \qquad\qquad \underline{Z} \approx \arctan\frac{\varepsilon''}{2\varepsilon'}\,. \tag{127}$$

Thus the attenuation is very small, and $\underline{\vec{E}}$ and $\underline{\vec{H}}$ are nearly in phase. The wave is almost the same as in the loss-free dielectric case. For example, in polystyrene a 10 MHz wave is attenuated only 0.5% per kilometre, and the phase difference between $\underline{\vec{E}}$ and $\underline{\vec{H}}$ is only 0.003°.

In the high-loss case with $\kappa \neq 0$ we have $\underline{\varepsilon} \approx -i\varepsilon'' = -i\dfrac{\kappa}{\omega}$ and

$$\sqrt{\underline{\varepsilon}} \approx \sqrt{\frac{\kappa}{2\omega}}\,(1-i) \quad \Rightarrow \qquad \alpha \approx \sqrt{\frac{\omega\mu\kappa}{2}} \qquad\qquad \beta \approx \sqrt{\frac{\omega\mu\kappa}{2}} \tag{128}$$

$$|\underline{Z}| \approx \sqrt{\frac{\omega\mu}{\kappa}} \qquad\qquad \underline{Z} \approx \frac{\pi}{4}\,. \tag{129}$$

Thus the attenuation is very large, and $\underline{\vec{H}}$ lags behind $\underline{\vec{E}}$ by 45°. The intrinsic impedance of a good conductor is extremely small at radio frequencies having a magnitude of $1.16 \cdot 10^{-3}\,\Omega$ for copper at 10 MHz. The wavelength is also very small compared to the free-space wavelength. For example, at 10 MHz the free-space wavelength is 30 m, while in copper the wavelength is only 0.131 mm. The attenuation in a good conductor is very rapid. For the 10 MHz wave mentioned above in copper the attenuation is 99.81% in 0.131 mm of travel. Thus waves do not penetrate metals very deeply. A metal acts as a shield against electromagnetic waves.

A wave starting at the surface of a good conductor and propagating inward is very quickly damped to insignificant values. The field is localized in a thin surface layer, this phenomenon being known as 'skin effect'. The distance over which a wave is attenuated to $1/e$ (36.8%) of its initial value is called the skin depth or depth of penetration δ. It is defined by $\alpha\delta = 1$, or

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\kappa}}\,. \tag{130}$$

The skin depth is very small for good conductors at radio frequencies, for example for copper at 10 MHz it is only 0.021 mm.

2.3 Energy density, energy flow, Poynting vector

2.3.1 The Poynting vector

The space containing a time-varying electromagnetic field is a carrier of electric and magnetic energy. According to Eqs. (87) and (89) the energy density inside a linear medium (i.e. a medium with constant permittivity and permeability independent of the field intensities) can be written as

$$w = w_e + w_m = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}) \ . \tag{131}$$

Thus, the total stored energy inside the volume V is

$$W = \frac{1}{2}\int\limits_V (\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\,\mathrm{d}V \ . \tag{132}$$

The time variation of the energy is given by

$$\frac{dW}{dt} = \frac{1}{2}\int\limits_V \frac{\partial}{\partial t}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\,\mathrm{d}V \ . \tag{133}$$

A scalar product can be differentiated using the product law. So from the first term we get

$$\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D}) = \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \frac{\partial\vec{E}}{\partial t}\cdot\vec{D} = 2\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \ . \tag{134}$$

In this equation the two terms of the sum can be reduced to a single term under the additional assumption that the dielectric is isotropic, i.e. ε is a scalar value independent of the orientation of the fields. Under the same assumption for the permeability μ the time variation of the energy gives

$$\frac{dW}{dt} = \int\limits_V (\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t})\,\mathrm{d}V \ . \tag{135}$$

Substituting Maxwell's equations

$$\begin{aligned} \frac{\partial \vec{B}}{\partial t} &= -\mathrm{curl}\,\vec{E}\,, & (136) \\ \frac{\partial \vec{D}}{\partial t} &= \mathrm{curl}\,\vec{H} - \vec{J} & (137) \end{aligned}$$

in Eq. (135), we get

$$\begin{aligned} \frac{dW}{dt} &= \int_V [\vec{E} \cdot (\mathrm{curl}\,\vec{H} - \vec{J}) + \vec{H} \cdot (-\mathrm{curl}\,\vec{E})]\,\mathrm{d}V & (138) \\ &= \int_V [\vec{E} \cdot \mathrm{curl}\,\vec{H} - \vec{H} \cdot \mathrm{curl}\,\vec{E} - \vec{E} \cdot \vec{J}]\,\mathrm{d}V\,. & (139) \end{aligned}$$

Using the vector identity

$$\mathrm{div}\,(\vec{A} \times \vec{B}) = \vec{B} \cdot \mathrm{curl}\,\vec{A} - \vec{A} \cdot \mathrm{curl}\,\vec{B} \tag{140}$$

leads to

$$\frac{dW}{dt} = \int_V \left(-\mathrm{div}\,(\vec{E} \times \vec{H}) - \vec{E} \cdot \vec{J} \right)\,\mathrm{d}V\,. \tag{141}$$

Applying Gauss' law we finally get the following equation, known as Poynting's law (see also Fig. 18):

$$\boxed{\frac{dW}{dt} = -\int_{\partial V} (\vec{E} \times \vec{H}) \cdot \mathrm{d}\vec{A} - \int_V (\vec{E} \cdot \vec{J})\,\mathrm{d}V\,.} \tag{142}$$

It states that the change of the electromagnetic energy inside a volume V can have two causes: (i) energy in the form of electromagnetic radiation is flowing through the boundary surface ∂V of the volume, (ii) an amount of the electromagnetic energy contained inside the volume is being converted into another form of energy or vice versa.

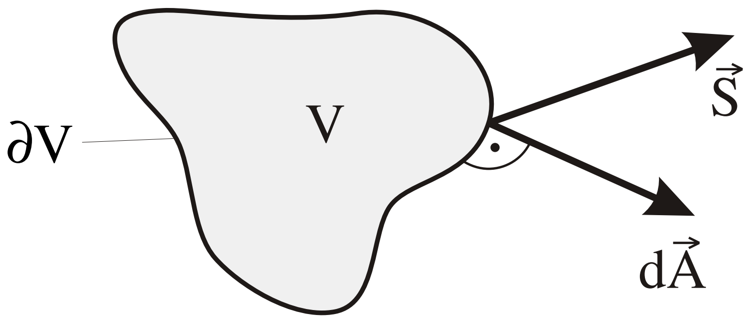

Fig. 18: Integration volume V and its boundary ∂V in Poynting's law

The Poynting vector

$$\vec{S} = \vec{E} \times \vec{H} \tag{143}$$

is defined as the cross product of the electric and magnetic field intensity. At every point of the space it points in the direction in which the electromagnetic energy flows. The integral of this vector throughout an arbitrary surface defines the amount of energy per second penetrating through this surface. The Poynting vector can thus be interpreted as a surface density of the energy flow. The terms energy flow vector or radiation vector are also commonly used.

The second term in Poynting's law describes the transformation of electromagnetic energy from or into another form of energy. In the general case the current density can be written

$$\vec{J} = \kappa \vec{E} + \vec{J}_i \,, \tag{144}$$

where $\vec{J}_i$ denotes the impressed-current density. Using these notations the second term of Poynting's law yields

$$\int_V (\vec{E} \cdot \vec{J})\, \mathrm{d}V = \int_V \kappa\, |\vec{E}|^2\, \mathrm{d}V + \int_V (\vec{E} \cdot \vec{J}_i)\, \mathrm{d}V \,. \tag{145}$$

The volume integral of $\kappa\, |\vec{E}|^2$ is always positive. Representing Joulean heat losses, it leads to a dissipation of the electromagnetic energy. The second term, the volume integral of $\vec{E} \cdot \vec{J}_i$, can also take negative values (if $\vec{E}$ and $\vec{J}_i$ are not in the same direction). In that case it can lead to an increase of the electromagnetic energy; an example is the radiation from an externally driven antenna.

2.3.2 *The complex Poynting vector*

When the time variation of the field is harmonic with angular frequency ω, again the complex notation of Maxwell's equations can be used:

$$\begin{aligned} \operatorname{curl} \underline{\vec{E}} &= -i\omega\underline{\mu}\underline{\vec{H}} \,, && (146)\\ \operatorname{curl} \underline{\vec{H}} &= i\omega\underline{\varepsilon}\, \underline{\vec{E}} \,, && (147) \end{aligned}$$

with generally complex material quantities

$$\underline{\varepsilon} = \varepsilon\, (1 - i \tan \delta_\varepsilon) \quad \text{and} \quad \underline{\mu} = \mu\, (1 - i \tan \delta_\mu) \,. \tag{148}$$

To derive a modified form of Poynting's law for the phasors $\underline{\vec{E}}$ and $\underline{\vec{H}}$, Eq. (146) is left-multiplied by $\underline{\vec{H}}^*$ (the conjugate value of $\underline{\vec{H}}$):

$$\underline{\vec{H}}^* \cdot \operatorname{curl} \underline{\vec{E}} = -i\omega\underline{\mu}\underline{\vec{H}} \cdot \underline{\vec{H}}^* \,. \tag{149}$$

Left-multiplication of the conjugate form of Eq. (147) by $\underline{\vec{E}}$ yields

$$\underline{\vec{E}} \cdot \operatorname{curl} \underline{\vec{H}}^* = -i\omega\underline{\varepsilon}^*\, \underline{\vec{E}} \cdot \underline{\vec{E}}^* \,. \tag{150}$$

Using the vector identity (140) again, we get

$$\begin{aligned} \operatorname{div} (\underline{\vec{E}} \times \underline{\vec{H}}^*) &= \underline{\vec{H}}^* \cdot \operatorname{curl} \underline{\vec{E}} - \underline{\vec{E}} \cdot \operatorname{curl} \underline{\vec{H}}^* \\ &= -i\omega\underline{\mu}\underline{\vec{H}} \cdot \underline{\vec{H}}^* + i\omega\underline{\varepsilon}^*\underline{\vec{E}} \cdot \vec{E}^* \,. && (151) \end{aligned}$$

The Poynting vector according to its definition (143) can be written

$$\vec{S}(t) = \mathcal{R}\mathrm{e}\{\underline{\mathrm{E}}\, \mathrm{e}^{\mathrm{i}\omega \mathrm{t}}\} \times \mathcal{R}\mathrm{e}\{\underline{\mathrm{H}}\, \mathrm{e}^{\mathrm{i}\omega \mathrm{t}}\} \,. \tag{152}$$

In most cases we are interested in determining the time average of the energy flux density $\overline{\vec{S}}$ rather than its instantaneous value $\vec{S}(t)$. Thus we integrate Eq. (152) over a period $T = 2\pi/\omega$ and divide it by the period

$$\begin{aligned} \overline{\vec{S}} &= \frac{1}{T} \int_0^T \mathrm{Re}\{\underline{\vec{E}}\, e^{i\omega t}\} \times \mathrm{Re}\{\underline{\vec{H}}\, e^{i\omega t}\}\, \mathrm{d}t \\ &= \ldots = \mathrm{Re}\left\{\frac{1}{2}\left(\underline{\vec{E}} \times \underline{\vec{H}}^*\right)\right\} \,. && (153) \end{aligned}$$

The vector

$$\underline{\vec{S}} := \frac{1}{2}\left(\underline{\vec{E}} \times \underline{\vec{H}}^*\right) \tag{154}$$

is called the complex Poynting vector. Its real part is equal to the time average of the energy flux density.

The complex form of Poynting's law is finally derived using Eq. (151):

$$\begin{aligned}
\int_{\partial V} \underline{\vec{S}} \cdot \mathrm{d}\vec{A} &= -2i\omega \int_V \left(\frac{\underline{\mu}|\underline{\vec{H}}|^2}{4} - \frac{\underline{\varepsilon}^*|\underline{\vec{E}}|^2}{4}\right) \mathrm{d}V \\
&= -2\omega \tan\delta_\mu \int_V \frac{\mu}{4}|\underline{\vec{H}}|^2 \mathrm{d}V \;-\; 2\omega \tan\delta_\varepsilon \int_V \frac{\varepsilon}{4}|\underline{\vec{E}}|^2 \mathrm{d}V \\
&\quad -2i\omega\Big(\int_V \frac{\mu}{4}|\underline{\vec{H}}|^2 \mathrm{d}V - \int_V \frac{\varepsilon}{4}|\underline{\vec{E}}|^2 \mathrm{d}V\Big) \\
&= -N_W - iN_B = -N_W - 2i\omega(W_m - W_e)\ .
\end{aligned} \tag{155}$$

The active power N_W represents the time-averaged Joulean heat produced inside the integration volume V, and the reactive power N_B represents the difference between the time-averaged magnetic and electric energy stored inside the volume.

For good conductors (metal) we have $W_e \ll W_m$ and

$$\begin{aligned}
\int_{\partial V} \underline{\vec{S}} \cdot \mathrm{d}\vec{A} &\approx -\int_V \frac{1}{2}\kappa|\underline{\vec{E}}|^2 \mathrm{d}V - i2\omega W_m \\
&= -N_W - iN_B\ .
\end{aligned} \tag{156}$$

This relationship enables us to calculate the resistance R_W and the internal inductance L_i of the conductor, especially in high frequencies. Defining complex current and voltage quantities as

$$\underline{I}^* = \oint \underline{\vec{H}}^* \cdot \mathrm{d}\vec{s} \tag{157}$$

and

$$\begin{aligned}
\underline{U} &= \int_1^2 \underline{\vec{E}} \cdot \mathrm{d}\vec{l} \\
&= \underline{I}\,(R_W + i\omega L_i) = \underline{I}\,\underline{Z}\ ,
\end{aligned} \tag{158}$$

N_W and N_B are given by

$$\begin{aligned}
N_W &= \frac{1}{2}|\underline{I}|^2 R_W\ , \\
N_B &= \frac{1}{2}|\underline{I}|^2 \omega L_i\ .
\end{aligned} \tag{159}$$

2.3.3 Phase, group and energy velocity

The velocity at which an equiphase surface travels is called the phase velocity of the wave. An equiphase plane $x_p(t)$ of a plane wave is defined by

$$\omega t - \beta x_p(t) = \text{constant}\ , \tag{160}$$

as we can see in Eq. (123). To derive the phase velocity of the wave, we differentiate this equation,

$$\omega - \beta \frac{dx_p}{dt} = 0\ , \tag{161}$$

and obtain

$$v_p = \frac{\omega}{\beta} = \lambda f \ . \tag{162}$$

In a non-conducting medium v_p is given by

$$v_p = \frac{1}{\sqrt{\varepsilon\mu}} \ . \tag{163}$$

As we know from wireless communications for the transmission of signals, wave trains rather than single monochromatic waves are used. In this case the velocity at which the signal travels is not the phase velocity but the group velocity v_g. The simplest case is one wave train consisting of two different sinusoidal waves of the same amplitude, but with slightly different frequencies $\omega_1 = \omega$ and $\omega_2 = \omega + \Delta\omega$ where $\Delta\omega \ll \omega$:

$$f(x,t) = \hat{f}\ \sin(\omega\, t - \beta\, x) + \hat{f}\ \sin([\omega + \Delta\omega]\, t - [\beta + \Delta\beta]\, x) \ . \tag{164}$$

The superposition of the two waves creates a beat (see Fig. 19) with an envelope propagating without any significant change in shape at the group velocity v_p.

Assuming $\Delta\omega \ll \omega$ and $\Delta\beta \ll \beta$, the resulting wave is given by

$$f(x,t) = 2\ \hat{f}\ \underbrace{\sin(\omega\, t - \beta\, x)}_{\text{carrier}}\ \underbrace{\cos(\frac{1}{2}[\Delta\omega\, t - \Delta\beta\, x])}_{\text{signal information}} \ . \tag{165}$$

While the high-frequency component (carrier) travels at the phase velocity

$$v_p = \frac{\omega}{\beta} \ , \tag{166}$$

the envelope (signal) travels with the group velocity

$$v_g = \frac{\Delta\omega}{\Delta\beta} \to \frac{d\omega}{d\beta} \ . \tag{167}$$

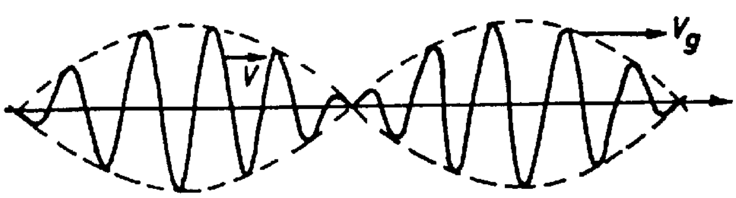

Fig. 19: The propagation of the envelope (signals carrier) of a wave train consisting of two waves with slightly differing frequencies

The energy stored in the electromagnetic fields of a wave propagates through space at a velocity called energy velocity v_E. To derive (one component of) this velocity we consider the small volume in Fig. 20.

The energy penetrating through the face ΔA_x during the time Δt is given by

$$N_x\, \Delta t = \Delta A_x\ S_x\ \Delta t \ , \tag{168}$$

where S_x is the x-component of the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$. It is equivalent to the energy stored in the volume $\Delta A_x\, \Delta x$, with $\Delta x = v_{E,x}\, \Delta t$:

$$W = W'\, \Delta x = w\ \Delta A\, \Delta x \ , \tag{169}$$

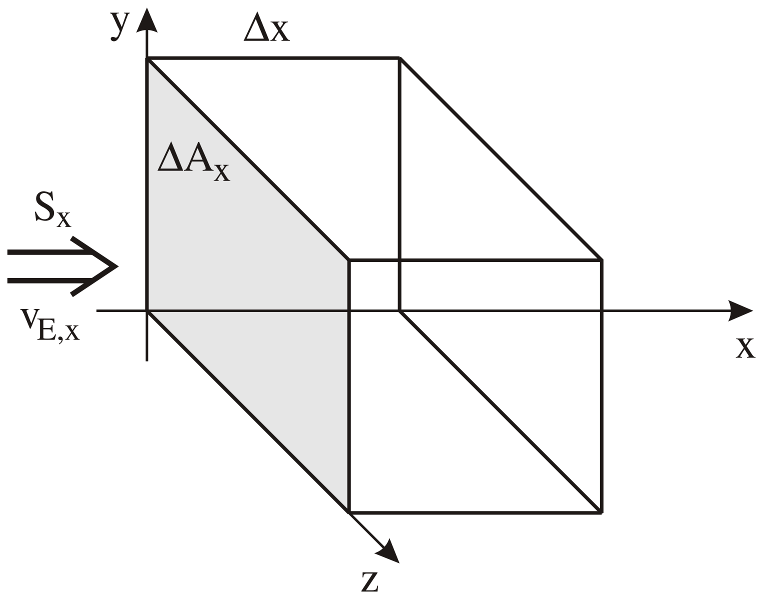

Fig. 20: Derivation of the x-component $v_{E,x}$ of the energy velocity of an electromagnetic wave

where the energy density w includes both the electric w_e and the magnetic w_m energy density.

For the energy velocity we obtain

$$N_x \, \Delta t = W' \, \Delta x \qquad \Rightarrow \quad v_{E,x} = \frac{N_x}{W'} = \frac{S_x}{w} \ . \tag{170}$$

In the general case we get the velocity field

$$\vec{v}_E(\vec{r}, t) = \frac{\vec{S}(\vec{r}, t)}{w(\vec{r}, t)} \ . \tag{171}$$

In most time-harmonic applications we are interested in the time-averaged quantity

$$\vec{v}_{E,\omega}(\vec{r}) = \frac{\mathrm{Re}\{\underline{\vec{S}}(\vec{r})\}}{\underline{w}(\vec{r})} \ . \tag{172}$$

For time-harmonic plane waves propagating in the x-direction in a loss-free medium,

$$\underline{\vec{E}} = \underline{E}_y \, \vec{e}_y \, e^{-i\beta x} \ , \qquad \underline{\vec{H}} = \frac{\underline{E}_y}{Z} \, \vec{e}_z \, e^{-i\beta x} \ , \tag{173}$$

we have

$$\underline{\vec{S}} \ = \ \frac{1}{2}(\underline{\vec{E}} \times \underline{\vec{H}}^*) = \frac{|\underline{E}_y|^2}{2Z} \, \vec{e}_x \ , \tag{174}$$

$$\underline{w} \ = \ \frac{1}{4}\varepsilon|\underline{E}_y|^2 + \frac{1}{4}\mu\frac{|\underline{E}_y|^2}{Z^2} = \frac{1}{2}\varepsilon|\underline{E}_y|^2 \ . \tag{175}$$

For the energy velocity we obtain

$$v_{E,\omega} = \frac{|\underline{E}_y|^2}{Z\varepsilon|\underline{E}_y|^2} = \frac{1}{\sqrt{\varepsilon\mu}} = c \ . \tag{176}$$

We find that in this case the energy velocity is identical to the phase velocity v_p.

For monochromatic plane waves all velocity definitions lead to the same value, i.e. we have $v_E = v_g = v_p$. This result, however, is *not* valid for general waves. The group velocity as well as the phase velocity may be larger than c, and in some cases the group velocity can even be negative. The energy velocity, however, is always bounded by $v_E \leq c$, ensuring the principle of causality. In waveguides we usually have

$$v_g = v_E < c \quad \text{and} \quad v_p > c \,. \tag{177}$$

3 BASIC THEORY OF WAVEGUIDES

3.1 Rectangular waveguides with perfectly conducting walls

3.1.1 The wave equation for waveguides (vector potential)

In this section the wave propagation in a rectangular waveguide is investigated. The guide should possess perfectly conducting walls and should be positioned (see Fig. 21) so the wave propagates in the $\pm z$-direction. The medium in the guide should be homogeneous, isotropic and linear. Assuming time-

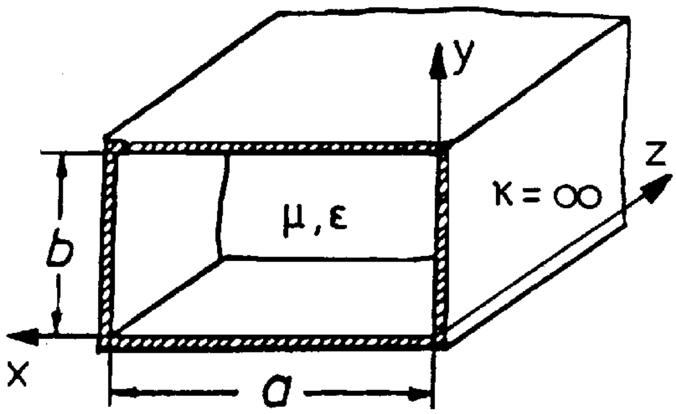

Fig. 21: Position of the rectangular waveguide in a Cartesian coordinate system

harmonic field quantities ($\sim e^{i\omega t}$) and, furthermore, the absence of space charges, Maxwell's equations result in

$$\begin{aligned} \operatorname{curl} \underline{\vec{E}} &= -i\omega\mu \underline{\vec{H}} \,, & (178)\\ \operatorname{curl} \underline{\vec{H}} &= i\omega\varepsilon \underline{\vec{E}} \,, & (179)\\ \operatorname{div} \underline{\vec{E}} &= 0 \,, & (180)\\ \operatorname{div} \underline{\vec{H}} &= 0 \,. & (181) \end{aligned}$$

It can be shown that two fundamental forms of five-component waves can exist in a waveguide. On the one hand, there are E waves with one vanishing component of the magnetic field (e.g. $\underline{H}_z = 0, \underline{E}_z \neq 0$); on the other hand, there are H waves with one vanishing component of the electric field (e.g. $\underline{E}_z = 0, \underline{H}_z \neq 0$). Hence, Maxwell's equations lead to two different classes of solutions. The general solution is obtained by a superposition of E and H waves.

To solve Eq. (178)–(181), a vector potential $\vec{A}$ is introduced. Hereby two approaches are possible:

$$\underline{\vec{E}} := \operatorname{curl} \underline{\vec{A}}^H \quad (182) \qquad \Big| \qquad \underline{\vec{H}} := \operatorname{curl} \underline{\vec{A}}^E \,. \quad (183)$$

The choice of the indices becomes more obvious later on.

This approach yields with Eq. (178):

$$\begin{aligned}\operatorname{curl}\underline{\vec{E}} &= \operatorname{curl}\operatorname{curl}\underline{\vec{A}}^H \\ &= \operatorname{grad}\operatorname{div}\underline{\vec{A}}^H - \Delta\underline{\vec{A}}^H \\ &= -i\omega\mu\,\underline{\vec{H}}.\end{aligned} \tag{184}$$

Insertion in Eq. (179) yields:

$$\begin{aligned}\operatorname{curl}\underline{\vec{H}} &= i\omega\varepsilon\operatorname{curl}\underline{\vec{A}}^H \\ \Rightarrow\ \underline{\vec{H}} &= i\omega\varepsilon\,\underline{\vec{A}}^H + \operatorname{grad}\underline{\Phi}^H,\end{aligned} \tag{185}$$

where $\operatorname{grad}\underline{\Phi}^H$ is an undefined constant of integration.
Inserting the last relation in Eq. (184) leads to:

$$\operatorname{grad}\operatorname{div}\underline{\vec{A}}^H - \Delta\underline{\vec{A}}^H = \omega^2\mu\varepsilon\underline{\vec{A}}^H - i\omega\mu\operatorname{grad}\underline{\Phi}^H.$$

According to the Lorentz convention $\underline{\Phi}^H$ is chosen as follows:

$$\operatorname{div}\underline{\vec{A}}^H = -i\omega\mu\,\underline{\Phi}^H\ . \tag{186}$$

This approach yields with Eq. (179):

$$\begin{aligned}\operatorname{curl}\underline{\vec{H}} &= \operatorname{curl}\operatorname{curl}\underline{\vec{A}}^E \\ &= \operatorname{grad}\operatorname{div}\underline{\vec{A}}^E - \Delta\underline{\vec{A}}^E \\ &= i\omega\varepsilon\,\underline{\vec{E}}.\end{aligned} \tag{187}$$

Insertion in Eq. (178) yields:

$$\begin{aligned}\operatorname{curl}\underline{\vec{E}} &= -i\omega\mu\operatorname{curl}\underline{\vec{A}}^E \\ \Rightarrow\ \underline{\vec{E}} &= -i\omega\mu\,\underline{\vec{A}}^E - \operatorname{grad}\underline{\Phi}^E\ ,\end{aligned} \tag{188}$$

where $\operatorname{grad}\underline{\Phi}^E$ is an undefined constant of integration.
Inserting the last relation in Eq. (187) leads to:

$$\operatorname{grad}\operatorname{div}\underline{\vec{A}}^E - \Delta\underline{\vec{A}}^E = \omega^2\mu\varepsilon\underline{\vec{A}}^E - i\omega\varepsilon\operatorname{grad}\underline{\Phi}^E\ .$$

According to the Lorentz convention $\underline{\Phi}^E$ is chosen as follows:

$$\operatorname{div}\underline{\vec{A}}^E = -i\omega\varepsilon\,\underline{\Phi}^E\ . \tag{189}$$

For both cases the same coordinate system independent vector wave equation is obtained:

$$\boxed{\Delta\underline{\vec{A}} + k^2\underline{\vec{A}} = 0} \tag{190}$$

with the wavenumber k:

$$\boxed{k^2 = \omega^2\mu\varepsilon\ .} \tag{191}$$

3.1.2 Solutions of the wave equation

In Cartesian coordinates the vector potential may have the components A_x, A_y, and A_z. It can be shown, however, that a vector potential with only one component is sufficient to describe all wave solutions:

$$\underline{\vec{A}}(x,y,z) = \underline{A}_z(x,y,z)\,\vec{e}_z\ . \tag{192}$$

Thus from the vector wave equation (190) the scalar wave equation, the so-called Helmholtz equation, can be obtained:

$$\frac{\partial^2\underline{A}_z}{\partial x^2} + \frac{\partial^2\underline{A}_z}{\partial y^2} + \frac{\partial^2\underline{A}_z}{\partial z^2} + k^2\underline{A}_z = 0\ . \tag{193}$$

Taking a Bernoulli ansatz in a product form,

$$\underline{A}_z(x,y,z) = \underline{f}(x)\,\underline{g}(y)\,\underline{h}(z)\ , \tag{194}$$

Eq. (193) yields

$$\underbrace{\frac{1}{\underline{f}}\frac{d^2\underline{f}}{dx^2}}_{=-k_x^2} + \underbrace{\frac{1}{\underline{g}}\frac{d^2\underline{g}}{dy^2}}_{=-k_y^2} + \underbrace{\frac{1}{\underline{h}}\frac{d^2\underline{h}}{dz^2}}_{=-k_z^2} + k^2 = 0\ . \tag{195}$$

To fulfil this differential equation for all x, y, z, the addends must be constants (real or complex) and must satisfy the dispersion relation

$$k_x^2 + k_y^2 + k_z^2 = k^2. \tag{196}$$

Appropriate solution functions for a waveguide extending in the z-direction are given by

$$\begin{aligned}
\underline{f}(x) &= \underline{A}_1 \cos k_x x + \underline{A}_2 \sin k_x x \ , && (197)\\
\underline{g}(y) &= \underline{B}_1 \cos k_y y + \underline{B}_2 \sin k_y y \ , && (198)\\
\underline{h}(z) &= \underline{C}_1\, e^{+ik_z z} + \underline{C}_2\, e^{-ik_z z} \ . && (199)
\end{aligned}$$

Introducing the notation

$$\left\{ \begin{array}{c} \cos k_x x \\ \sin k_x x \end{array} \right\} = \underline{A}_1 \cos k_x x + \underline{A}_2 \sin k_x x \tag{200}$$

the general solution for the vector potential is

$$\underline{A}_z(x,y,z) = \left\{ \begin{array}{c} \cos k_x x \\ \sin k_x x \end{array} \right\} \left\{ \begin{array}{c} \cos k_y y \\ \sin k_y y \end{array} \right\} \left\{ \begin{array}{c} e^{+ik_z z} \\ e^{-ik_z z} \end{array} \right\} . \tag{201}$$

Equation (201) is the solution for the vector potential for both H and E waves. The field components derived from this solution, however, differ depending on whether ansatz (182) or Eq. (183) is chosen:

a) H **waves**:

$$\underline{\vec{A}} = \underline{\vec{A}}^H \;\Rightarrow\; \underline{\vec{E}} = \mathrm{curl}\, \underline{\vec{A}} \tag{202}$$

Using this approach, H waves with $\underline{H}_z \neq 0$ are received. As these waves only have transversal components of the electric field, they are also referred to as TE waves.

The electric-field components of H waves are implied by the approach above, while the magnetic-field components can be derived from the electric field using Faraday's law Eq. (178):

$$\underline{\vec{H}} = -\frac{1}{i\omega\mu}\mathrm{curl}\, \underline{\vec{E}} = -\frac{1}{i\omega\mu}\mathrm{curl}\,\mathrm{curl}\, \underline{\vec{A}} \ . \tag{203}$$

In the next equations, first the general derivation of all electric- and magnetic-field components is given, and then — in the last column — the explicit expressions assuming the vector potential in Eq. (201). Applying the upper sign of the exponential function, a wave propagation in the $-z$-direction, and otherwise in the $+z$-direction, is described.

$$\begin{aligned}
\underline{E}_x &= +\frac{\partial \underline{A}_z}{\partial y} &&= \underline{C}\, k_y \left\{ \begin{array}{c} \sin k_x x \\ \cos k_x x \end{array} \right\} \left\{ \begin{array}{c} \cos k_y y \\ -\sin k_y y \end{array} \right\} e^{\pm ik_z z} \ , \\
\underline{E}_y &= -\frac{\partial \underline{A}_z}{\partial x} &&= \underline{C}\, k_x \left\{ \begin{array}{c} -\cos k_x x \\ \sin k_x x \end{array} \right\} \left\{ \begin{array}{c} \sin k_y y \\ \cos k_y y \end{array} \right\} e^{\pm ik_z z} \ , \\
\underline{H}_x &= -\frac{1}{i\omega\mu}\frac{\partial^2 \underline{A}_z}{\partial x\, \partial z} &&= \mp\underline{C}\frac{k_x k_z}{\omega\mu} \left\{ \begin{array}{c} \cos k_x x \\ -\sin k_x x \end{array} \right\} \left\{ \begin{array}{c} \sin k_y y \\ \cos k_y y \end{array} \right\} e^{\pm ik_z z} \ , \qquad (204)\\
\underline{H}_y &= -\frac{1}{i\omega\mu}\frac{\partial^2 \underline{A}_z}{\partial y\, \partial z} &&= \mp\underline{C}\frac{k_y k_z}{\omega\mu} \left\{ \begin{array}{c} \sin k_x x \\ \cos k_x x \end{array} \right\} \left\{ \begin{array}{c} \cos k_y y \\ -\sin k_y y \end{array} \right\} e^{\pm ik_z z} \ , \\
\underline{H}_z &= -\frac{k^2 - k_z^2}{i\omega\mu}\underline{A}_z &&= -\underline{C}\,\frac{k^2 - k_z^2}{i\omega\mu} \left\{ \begin{array}{c} \sin k_x x \\ \cos k_x x \end{array} \right\} \left\{ \begin{array}{c} \sin k_y y \\ \cos k_y y \end{array} \right\} e^{\pm ik_z z} \ .
\end{aligned}$$

b) E **waves**:

$$\underline{\vec{A}} = \underline{\vec{A}}^E \;\Rightarrow\; \underline{\vec{H}} = \text{curl}\,\underline{\vec{A}}\ . \tag{205}$$

Using this approach, E waves with $\underline{E}_z \neq 0$ are obtained, which only have transversal components of the magnetic field (TM waves). Here the magnetic-field components are implied by the potential approach above, while the electric components can be derived from the magnetic field using Ampere's law, Eq. (179):

$$\underline{\vec{E}} = \frac{1}{i\omega\varepsilon}\text{curl}\,\underline{\vec{H}} = \frac{1}{i\omega\varepsilon}\text{curl}\,\text{curl}\,\underline{\vec{A}}\,. \tag{206}$$

The five field components of the wave read as

$$\begin{aligned}
\underline{H}_x &= +\frac{\partial \underline{A}_z}{\partial y} &&= \underline{C}\,k_y \begin{Bmatrix} \sin k_x x \\ \cos k_x x \end{Bmatrix} \begin{Bmatrix} \cos k_y y \\ -\sin k_y y \end{Bmatrix} e^{\pm i k_z z}\,,\\
\underline{H}_y &= -\frac{\partial \underline{A}_z}{\partial x} &&= \underline{C}\,k_x \begin{Bmatrix} -\cos k_x x \\ \sin k_x x \end{Bmatrix} \begin{Bmatrix} \sin k_y y \\ \cos k_y y \end{Bmatrix} e^{\pm i k_z z}\,,\\
\underline{E}_x &= \frac{1}{i\omega\varepsilon}\frac{\partial^2 \underline{A}_z}{\partial x\,\partial z} &&= \pm\underline{C}\frac{k_x k_z}{\omega\varepsilon} \begin{Bmatrix} \cos k_x x \\ -\sin k_x x \end{Bmatrix} \begin{Bmatrix} \sin k_y y \\ \cos k_y y \end{Bmatrix} e^{\pm i k_z z}\,,\\
\underline{E}_y &= \frac{1}{i\omega\varepsilon}\frac{\partial^2 \underline{A}_z}{\partial y\,\partial z} &&= \pm\underline{C}\frac{k_y k_z}{\omega\varepsilon} \begin{Bmatrix} \sin k_x x \\ \cos k_x x \end{Bmatrix} \begin{Bmatrix} \cos k_y y \\ -\sin k_y y \end{Bmatrix} e^{\pm i k_z z}\,,\\
\underline{E}_z &= \frac{k^2 - k_z^2}{i\omega\varepsilon}\underline{A}_z &&= \underline{C}\,\frac{k^2 - k_z^2}{i\omega\varepsilon} \begin{Bmatrix} \sin k_x x \\ \cos k_x x \end{Bmatrix} \begin{Bmatrix} \sin k_y y \\ \cos k_y y \end{Bmatrix} e^{\pm i k_z z}\,.
\end{aligned} \tag{207}$$

The superposition of all H_z and E_z waves yields the general solution of Eqs. (178–181). Instead of the vector potential in the z-direction, a potential in the x- or y-direction can be chosen, also leading to the general solution. However, the approach of a potential in the direction of propagation is usually preferred in a rectangular waveguide.

3.1.3 Classification of waveguide modes

So far, for the solution of the vector wave equation the specific shape of the waveguide has not been exploited. This is done in the next step, imposing boundary conditions for the electric-field components. At the (perfectly conducting) walls of the rectangular guide in Fig. 21 the tangential components must vanish, leading to (for all z):

1) $\underline{E}_x(y=0) = \underline{E}_x(y=b) = 0 \qquad \forall x \in [0,a]\,,$
2) $\underline{E}_y(x=0) = \underline{E}_y(x=a) = 0 \qquad \forall y \in [0,b]\,,$
3) $\underline{E}_z(y=0) = \underline{E}_z(y=b) = 0 \qquad \forall x \in [0,a]\,,$
   $\underline{E}_z(x=0) = \underline{E}_z(x=a) = 0 \qquad \forall y \in [0,b]\,.$

A TE wave with $\underline{E}_x \sim \partial \underline{A}_z^H/\partial y$ and $\underline{E}_y \sim \partial \underline{A}_z^H/\partial x$ satisfies the first and second boundary conditions only if all sine-terms vanish in Eq. (201), i.e.

$$\underline{A}_z^H \sim \cos k_x x\ \cos k_y y\ .$$

This leads to (for all z)

$$\underline{E}_x \sim \cos k_x x\ \sin k_y y\ ,$$
$$\underline{E}_y \sim \sin k_x x\ \cos k_y y\ .$$

The boundary conditions, moreover, require

$$\sin k_x a = 0 \;\text{ and }\; \sin k_y b = 0 \;,$$

which can be satisfied by the eigenvalues

$$k_x = \frac{m\,\pi}{a} \;\text{ and }\; k_y = \frac{n\,\pi}{b} \;, \tag{208}$$

where $m, n = 0, 1, 2, \ldots$ except $m = n = 0$ (this would lead to a trivial solution with all fields equal to zero).

In the TM case the third boundary condition requires vanishing cosine-terms because of $\underline{E}_z \sim \underline{A}_z^E$, i.e.

$$\underline{A}_z^E \sim \sin k_x x \;\sin k_y y \;.$$

An additional requirement is

$$\sin k_x a = 0 \;\text{ and }\; \sin k_y b = 0 \;,$$

leading to the *same* eigenvalues as in the TE case

$$k_x = \frac{m\,\pi}{a} \;\text{ and }\; k_y = \frac{n\,\pi}{b} \;,$$

but now $m, n = 1, 2, ...$ excluding the trivial solution.

In both approaches the introduction of boundary conditions leads to an infinite number of discrete modes, which are distinguished by the parameters m and n. These modes are referred to by the notation

$$\text{TE}_{z\,mn} \quad \text{and} \quad \text{TM}_{z\,mn} \tag{209}$$

and have the following field components:
a) $\text{TE}_{z\,mn}$ modes (H_z waves):

$$\underline{E}_x = -\underline{C}^H \frac{n\,\pi}{b} \cos(\frac{m\,\pi}{a} x)\; \sin(\frac{n\,\pi}{b} y)\; e^{\pm i k_z z} \;, \tag{210}$$

$$\underline{E}_y = \underline{C}^H \frac{m\,\pi}{a} \sin(\frac{m\,\pi}{a} x)\; \cos(\frac{n\,\pi}{b} y)\; e^{\pm i k_z z} \;, \tag{211}$$

$$\underline{E}_z = 0 \;, \tag{212}$$

$$\underline{H}_x = \pm \frac{k_z}{\omega\mu} \underline{E}_y \;, \tag{213}$$

$$\underline{H}_y = \mp \frac{k_z}{\omega\mu} \underline{E}_x \;, \tag{214}$$

$$\underline{H}_z = -\underline{C}^H \frac{k^2 - k_z^2}{i\omega\mu} \cos(\frac{m\,\pi}{a} x)\; \cos(\frac{n\,\pi}{b} y)\; e^{\pm i k_z z} \;. \tag{215}$$

b) $\text{TM}_{z\,mn}$ modes (E_z waves):

$$\underline{H}_x = \underline{C}^E \frac{n\,\pi}{b} \sin(\frac{m\,\pi}{a} x)\; \cos(\frac{n\,\pi}{b} y)\; e^{\pm i k_z z} \;, \tag{216}$$

$$\underline{H}_y = -\underline{C}^E \frac{m\,\pi}{a} \cos(\frac{m\,\pi}{a} x)\; \sin(\frac{n\,\pi}{b} y)\; e^{\pm i k_z z} \;, \tag{217}$$

$$\underline{H}_z = 0 \;, \tag{218}$$

$$\underline{E}_x = \mp \frac{k_z}{\omega\varepsilon} \underline{H}_y \;, \tag{219}$$

$$\underline{E}_y = \pm \frac{k_z}{\omega\varepsilon} \underline{H}_x \;, \tag{220}$$

$$\underline{E}_z = \underline{C}^E \frac{k^2 - k_z^2}{i\omega\varepsilon} \sin(\frac{m\,\pi}{a} x)\; \sin(\frac{n\,\pi}{b} y)\; e^{\pm i k_z z} \;. \tag{221}$$

In a rectangular waveguide the wave impedance is defined by the quotient of the transversal components:

$$\underline{Z}_W = \frac{\underline{E}_T}{\underline{H}_T} \; . \tag{222}$$

Considering Eqs. (213) and (214) for TE waves or Eqs. (219) and (220) for TM waves, the wave impedance is

$$\underline{Z}_W = \begin{cases} \underline{Z}^H = \dfrac{\omega\mu}{k_z} & \text{TE modes} \\ \underline{Z}^E = \dfrac{k_z}{\omega\varepsilon} & \text{TM modes} \; . \end{cases} \tag{223}$$

The propagation characteristics of the waveguide modes are determined by the longitudinal wavenumber k_z, and depend on the transversal wavenumbers (eigenvalues) k_x and k_y according to the dispersion relation (196):

$$\begin{aligned} k_x^2 + k_y^2 + k_z^2 &= k^2 = \omega^2\mu\varepsilon \\ \Rightarrow \quad k_z^2 &= \omega^2\mu\varepsilon - (k_x^2 + k_y^2) = \frac{\omega^2}{c^2} - (k_x^2 + k_y^2) \; . \end{aligned} \tag{224}$$

For frequencies with $(\omega/c)^2 > k_x^2 + k_y^2$ the squared wavenumber k_z^2 is positive and we get a propagating wave:

$$\mathrm{Re}\{e^{ik_z z}\} = \cos(k_z z) \tag{225}$$

with a real k_z. For lower frequencies with $(\omega/c)^2 < k_x^2 + k_y^2$ the wavenumber becomes imaginary, $k_z = i\alpha$, and an exponential attenuation in the z-direction occurs:

$$\mathrm{Re}\{e^{ik_z z}\} = e^{-\alpha z} \; . \tag{226}$$

The limit between both cases defines the cutoff frequency of the waveguide mode:

$$f_c = \frac{c}{2\pi}\sqrt{k_x^2 + k_y^2} \; . \tag{227}$$

With $k_x = \frac{m\pi}{a}$ and $k_y = \frac{n\pi}{b}$ in the rectangular waveguide we get

$$f_{c\,mn} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{2\,a}\right)^2 + \left(\frac{n}{2\,b}\right)^2} . \tag{228}$$

Below this frequency the corresponding mode cannot propagate in the guide.

At frequencies above the cutoff frequency $f_{c\,mn}$, waves propagate corresponding to $e^{\pm ik_z z}$ in the positive or negative z-direction at a frequency f along the guide with the wavelength

$$\lambda_G = \frac{\lambda}{\sqrt{1 - (\frac{\lambda}{\lambda_c})^2}} \; , \tag{229}$$

where λ is the free-space and λ_c is the cutoff wavelength.

All modes in a waveguide can be sorted by referring to their cutoff frequency. In guides with $a > b$ (see Fig. 21) the mode with the lowest cutoff frequency is the TE_{10} mode. For example, for $a = 2 \cdot b$ it is the only propagating mode in the frequency range $f \in [f_{c,10},\ 2 \cdot f_{c,10}]$, and therefore the most important working mode in rectangular guides. The field pattern of this mode is shown in Fig. 22. The TM mode with the lowest cutoff frequency is the $E_{z\,11}$ mode.

As an example we consider a waveguide with

$$a = 6 \text{ cm}, \quad b = 3 \text{ cm} \; , \tag{230}$$

allowing modes with the following cutoff frequencies according to Eq. (228):

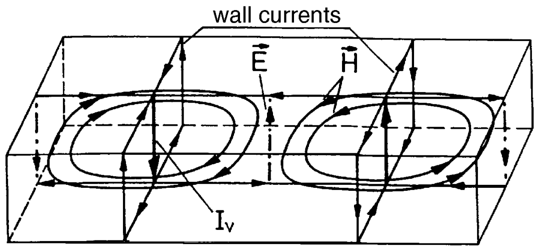

Fig. 22: $H_{z\ 10}$-wave in a rectangular waveguide

Type	m	n	f_c/GHz
TE	1	0	2.498
TE	0	1	4.997
TE	2	0	4.997
TE,TM	1	1	5.586
TE,TM	2	1	7.066
			...

Figure 23 shows the dispersion curves $k_z^2(f^2)$ (left) and $|k_z(f)|$ (right) of these modes. For frequencies below cutoff, k_z^2 becomes negative and k_z imaginary. For frequencies 2.498 GHz $< f <$ 4.997 GHz the TE_{10} mode is the only propagating mode in this waveguide.

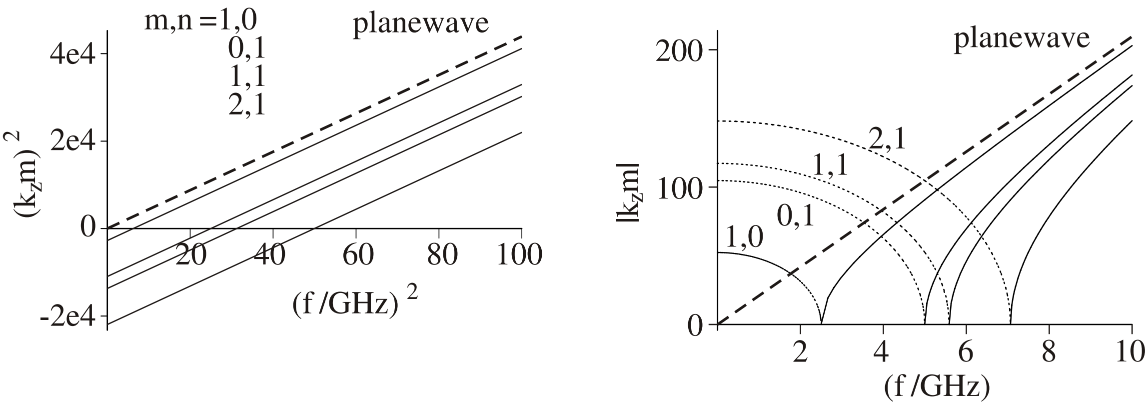

Fig. 23: Dispersion curves in a rectangular waveguide with $a = 6$ cm and $b = 3$ cm

Since $a = 2 \cdot b$ in this waveguide, the modes TE_{01} and TE_{20} are degenerate, i.e. they have identical cutoff frequencies and identical dispersion curves.

3.2 Other waveguides

3.2.1 Circular waveguides

By analogy with rectangular waveguides the approach

$$\vec{E} = \text{curl}\, A_z^H \vec{e}_z \quad (H \text{ waves}) \tag{231}$$

or

$$\vec{H} = \text{curl}\, A_z^E \vec{e}_z \quad (E \text{ waves}) \tag{232}$$

is chosen. Substituting into Maxwell's equations and enforcing the Lorentz gauge, the vector wave equation (190) and, with $\vec{A} = A\,\vec{e}_z$, the Helmholtz equation (193) are obtained.

Using the Δ operator for cylindrical coordinates in Eq. (193) yields

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \varphi^2} + \frac{\partial^2 A}{\partial z^2} + k^2 A = 0 \; . \tag{233}$$

Again an ansatz in product form,

$$A = f(r)\, g(\varphi)\, h(z) \; , \tag{234}$$

is used. Inserting this ansatz (234) into the Helmholtz equation (193) and dividing by A yields

$$\frac{1}{f}\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{1}{f}\frac{df}{dr} + \frac{1}{r^2}\underbrace{\frac{1}{g}\frac{d^2 g}{d\varphi^2}}_{=-\mu^2} + \underbrace{\frac{1}{h}\frac{d^2 h}{dz^2}}_{=-k_z^2} + k^2 = 0 \; . \tag{235}$$

The solutions for $g(\varphi)$ and $h(z)$ are obtained using the approaches

$$\frac{1}{g}\frac{d^2 g}{d\varphi^2} = -\mu^2 \; , \tag{236}$$

$$\frac{1}{h}\frac{d^2 h}{dz^2} = -k_z^2 \; , \tag{237}$$

and for $\mu \neq 0$ and $k_z \neq 0$ they can be written as

$$g(\varphi) = \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \; \mu\varphi \right\} \quad \text{or} \quad \left\{ e^{\pm i\mu\varphi} \right\} \tag{238}$$

and

$$h(z) = \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \; k_z z \right\} \quad \text{or} \quad \left\{ e^{\pm i k_z z} \right\} \; . \tag{239}$$

To obtain the function f, Eqs. (236) and (237) are inserted into the Helmholtz equation (193) and are multiplied by f:

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(k^2 - k_z^2 - \frac{\mu^2}{r^2} \right) f = 0 \; . \tag{240}$$

This Bessel differential equation has the general solution

$$f(r) = \mathrm{Z}_\mu\left(r\sqrt{k^2 - k_z^2} \right) \; . \tag{241}$$

Consequently the solution of the Helmholtz equation for the vector potential A in cylindrical coordinates reads as

$$A = \mathrm{Z}_\mu(Kr) e^{\pm i\mu\varphi} e^{\pm i k_z z} \qquad \text{or} \quad A = \mathrm{Z}_\mu(Kr) \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \; \mu\varphi \right\} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \; k_z z \right\} \tag{242}$$

with the dispersion equation

$$k^2 = K^2 + k_z^2 \; . \tag{243}$$

Z_μ denotes the *cylindrical functions*, which consist of Bessel and Neumann functions (see Fig. 24):

$$\mathrm{Z}_\mu(Kr) = A\mathrm{J}_\mu(Kr) + B\mathrm{N}_\mu(Kr). \tag{244}$$

A combination of the Bessel and Neumann functions yields the Hankel functions of the first and second kind

$$\mathrm{H}_\mu^{(1)}(Kr) = \mathrm{J}_\mu(Kr) + i\mathrm{N}_\mu(Kr), \tag{245}$$

$$\mathrm{H}_\mu^{(2)}(Kr) = \mathrm{J}_\mu(Kr) - i\mathrm{N}_\mu(Kr), \tag{246}$$

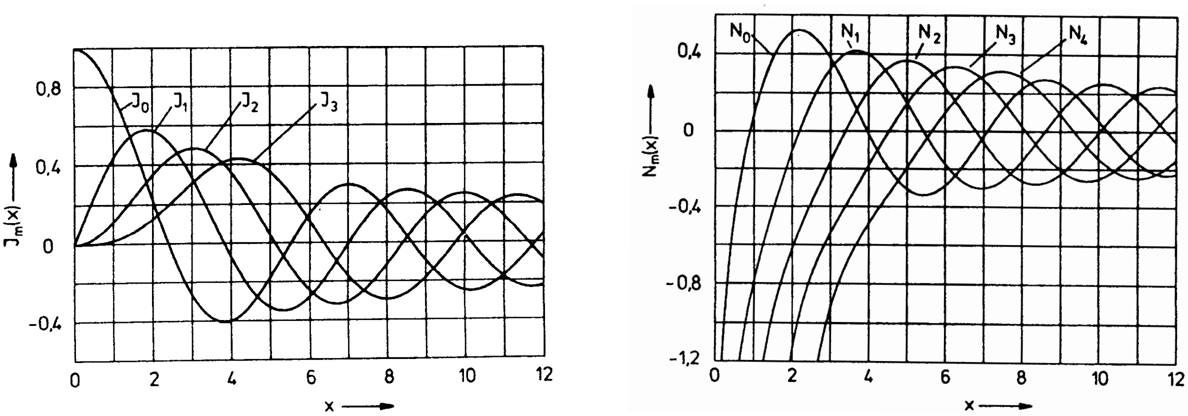

Fig. 24: The Bessel functions $\mathrm{J}_m(x)$ and Neumann functions $\mathrm{N}_m(x)$ with integer order m

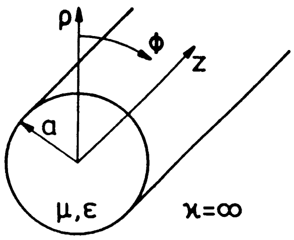

Fig. 25: Circular waveguide

which specify a wave propagating in the opposite direction to the Bessel and Neumann functions describing a standing wave in the radial direction.

To calculate the wave types in a cylindrical waveguide with a perfectly conducting wall the general solution is specialized. Because of the azimuthal periodicity of the configuration shown in Fig. 25, the condition

$$A(\varphi) = A(\varphi + 2\pi) \tag{247}$$

must hold. This leads to the relation

$$e^{\pm i\mu\varphi} = e^{\pm i\mu(\varphi+\pi)} , \tag{248}$$

which is satisfied by integer μ:

$$\mu = m = 0, 1, 2, 3, \ldots \tag{249}$$

The φ-dependency is described by a linear combination of sine- and cosine-functions. Because of the rotational symmetry of the configuration, the origin could be chosen such that only a sine- or a cosine-function is sufficient to describe the φ-dependency. Because the sine-function is permanently zero if $m = 0$, the cosine-function is chosen. Thus the vector potential is given by

$$A \sim \cos(m\varphi) . \tag{250}$$

The electromagnetic field must be finite for $r \to 0$ in the case of both wave forms, so the solution can only consist of Bessel functions (see Fig. 24):

$$A \sim \mathrm{J}_m(Kr) . \tag{251}$$

Consequently, for the vector potential of an electromagnetic wave propagating in the positive z-direction along a circular waveguide we have

$$\boxed{A = C\mathrm{J}_m(Kr)\ \cos m\varphi\, e^{-ik_z z} \quad \text{, where} \quad k_z = \sqrt{k^2 - K^2}\, .} \tag{252}$$

The field components for $E_{z\,mn}$ waves (TM waves) follow from the approach $\vec{H} = \mathrm{curl}\,\vec{A}$ and Eq. (252):

$$\begin{aligned}
H_r &= \frac{1}{r}\frac{\partial A_z}{\partial \varphi} &&= -\underline{C}^E \frac{m}{r}\, \mathrm{J}_m(Kr)\, \sin(m\varphi)\, e^{-ik_z z}\ , \\
H_\varphi &= -\frac{\partial A_z}{\partial r} &&= -\underline{C}^E K\, \mathrm{J}_m'(Kr)\, \cos(m\varphi)\, e^{-ik_z z}\ , \\
E_r &= -\frac{1}{i\omega\varepsilon}\frac{\partial H_\varphi}{\partial z} &&= -\underline{C}^E \frac{k_z}{\omega\varepsilon} K\, \mathrm{J}_m'(Kr)\, \cos(m\varphi)\, e^{-ik_z z}\ , \\
E_\varphi &= \frac{1}{i\omega\varepsilon}\frac{\partial H_r}{\partial z} &&= \underline{C}^E \frac{k_z}{\omega\varepsilon}\frac{m}{r}\, \mathrm{J}_m(Kr)\, \sin(m\varphi)\, e^{-ik_z z}\ , \\
E_z &= \frac{K^2}{i\omega\varepsilon} A_z &&= \underline{C}^E \frac{K^2}{i\omega\varepsilon}\, \mathrm{J}_m(Kr)\, \cos(m\varphi)\, e^{-ik_z z}\ ,
\end{aligned} \tag{253}$$

where Z_m' denotes the differentiation

$$\mathrm{Z}_m' = \frac{d\mathrm{Z}_m(Kr)}{d(Kr)}\ . \tag{254}$$

The boundary condition of vanishing E_φ and E_z at $r = a$ requires

$$\mathrm{J}_m(Kr) = 0 \quad \text{if} \quad r = a\ , \tag{255}$$

because E_φ and E_z are proportional to $\mathrm{J}_m(Kr)$. Thus the eigenvalue K is

$$K = \frac{j_{mn}}{a}\ , \tag{256}$$

where j_{mn} is the nth root of J_m. Numerical values for the roots can be found, for example, in Ref. [4]. With the eigenvalue K from Eq. (256), the cutoff frequency ω_c in a cylindrical waveguide is

$$\omega_{c\,mn} = \frac{j_{mn}}{a} c\ . \tag{257}$$

The field components for $H_{z\,mn}$ waves (TE waves) follow from the approach $\vec{E} = \mathrm{curl}\,\vec{A}$ and again Eq. (252):

$$\begin{aligned}
E_r &= \frac{1}{r}\frac{\partial A_z}{\partial \varphi} &&= -\underline{C}^H \frac{m}{r}\, \mathrm{J}_m(Kr)\, \sin(m\varphi)\, e^{-ik_z z}\ , \\
E_\varphi &= -\frac{\partial A_z}{\partial r} &&= -\underline{C}^H K\, \mathrm{J}_m'(Kr)\, \cos(m\varphi)\, e^{-ik_z z}\ , \\
H_r &= \frac{1}{i\omega\mu}\frac{\partial E_\varphi}{\partial z} &&= \underline{C}^H \frac{k_z}{\omega\mu} K\, \mathrm{J}_m'(Kr)\, \cos(m\varphi)\, e^{-ik_z z} \\
H_\varphi &= -\frac{1}{i\omega\mu}\frac{\partial E_r}{\partial z} &&= -\underline{C}^H \frac{k_z}{\omega\mu}\frac{m}{r}\, \mathrm{J}_m(Kr)\, \sin(m\varphi)\, e^{-ik_z z}\ , \\
H_z &= -\frac{K^2}{i\omega\mu} A_z &&= -\underline{C}^H \frac{K^2}{i\omega\mu}\, \mathrm{J}_m(Kr)\, \cos(m\varphi)\, e^{-ik_z z}\ .
\end{aligned} \tag{258}$$

Because of the proportionality of E_φ to J_m', in this case the boundary condition yields the eigenvalue

$$K = \frac{j_{mn}'}{a}\ , \tag{259}$$

where j_{mn}' is the nth root of J_m'. With this eigenvalue the cutoff frequency for a $H_{z\,mn}$ mode becomes

$$\omega_{c\,mn} = \frac{j_{mn}'}{a} c\ . \tag{260}$$

A comparison of the numerical values for the cutoff frequencies (257) and (260) in a waveguide with radius $a = 3$ cm yields the lowest cutoff frequency for the TE_{11} mode:

Type	m	n	f_c/GHz
TE	1	1	2.928
TM	0	1	3.825
TE	2	1	4.858
TE	0	1	6.094
TM	1	1	6.094
TE	3	1	6.682
			...

Again, some cutoff frequencies for E and H waves are identical, e.g. the TE_{01} and TM_{11} modes are degenerate. The field pattern of the TE_{11} mode is shown in Fig. 26. Note that there exist two TE_{11} modes with different polarization of the electric field.

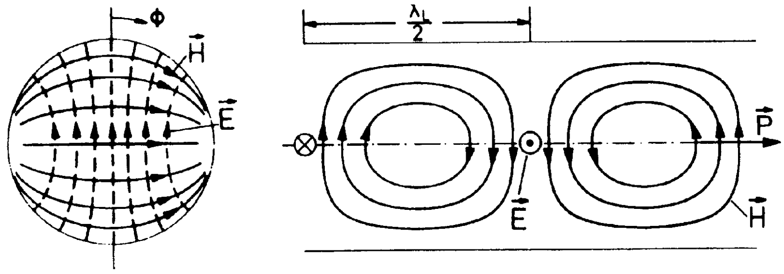

Fig. 26: Field lines of the $H_{z\,11}$ mode in a circular waveguide

3.2.2 *Coaxial waveguides*

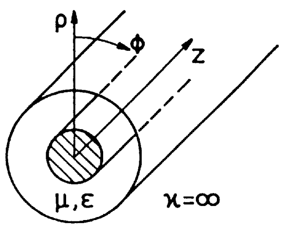

Fig. 27: A coaxial waveguide

In a coaxial waveguide, shown in Fig. 27 the z-axis is free of fields. Hence, for the general solution the cylindrical functions must contain Bessel functions as well as Neumann functions:

$$\mathrm{Z}_m(Kr) = A\mathrm{J}_m(Kr) + B\mathrm{N}_m(Kr)\ . \tag{261}$$

To determine the eigenvalue K the boundary conditions

$$E_z, E_\varphi = 0 \quad \text{if} \quad r = r_i, r_a \tag{262}$$

are imposed, leading to

$$\mathrm{Z}_m(Kr) \;=\; 0 \quad \text{if} \quad r = r_i, r_a \qquad (E \text{ waves})\ , \tag{263}$$

$$\mathrm{Z}'_m(Kr) \;=\; 0 \quad \text{if} \quad r = r_i, r_a \qquad (H \text{ waves})\ . \tag{264}$$

With these equations an eigenvalue equation

$$\frac{\mathrm{N}_m(Kr_i)}{\mathrm{J}_m(Kr_i)} = \frac{\mathrm{N}_m(Kr_a)}{\mathrm{J}_m(Kr_a)} \quad \text{with} \quad E \text{ waves ,} \tag{265}$$

$$\frac{\mathrm{N}'_m(Kr_i)}{\mathrm{J}'_m(Kr_i)} = \frac{\mathrm{N}'_m(Kr_a)}{\mathrm{J}'_m(Kr_a)} \quad \text{with} \quad H \text{ waves} \tag{266}$$

can be established, which usually has to be evaluated numerically. Figure 28 shows the field lines of the $H_{z\,11}$ mode, which has the lowest cutoff frequency (except for the TEM waves in the next paragraph).

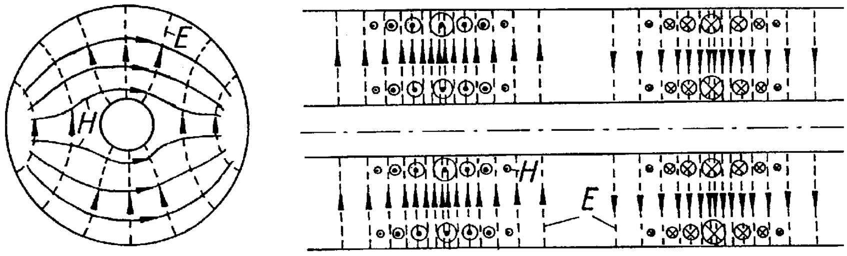

Fig. 28: Field lines of the $H_{z\,11}$ mode in a coaxial waveguide

Now we examine whether a wave exists in a coaxial waveguide with a cutoff frequency of zero. For the cutoff frequency, $k_z = 0$ applies and hence

$$\omega_c = \frac{1}{\mu\varepsilon} K \stackrel{!}{=} 0 \; ,$$

i.e. for a cutoff frequency of zero the eigenvalue K becomes

$$K = 0 \; . \tag{267}$$

The calculation of this wave starts with Eq. (240), reduced by $K = 0$ and $\mu = 0$ to

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} = 0 \; , \tag{268}$$

which has the solution

$$f = C_1 \ln \frac{r}{C_2} \; . \tag{269}$$

The vector potential can be written

$$A = C_1 \ln \frac{r}{C_2} \, e^{-ikz} \; , \tag{270}$$

where from $K = 0$ we have $k_z = k$. Note that this approach is not possible in a circular waveguide, because the ln-function tends toward infinity for $r \to 0$.

The field components are calculated using the approach $\vec{H} = \operatorname{curl} \vec{A}$:

$$\begin{aligned} E_r &= \frac{1}{r} E_0 \, e^{-ikz} \; , \\ H_\varphi &= \frac{1}{Z} E_r = \frac{1}{Z}\frac{1}{r} E_0 \, e^{-ikz} \; , \\ E_\varphi &= E_z = H_r = H_z = 0 \; , \end{aligned} \tag{271}$$

where $Z = \sqrt{\frac{\mu}{\varepsilon}}$. (Generally the approach $\vec{E} = \operatorname{curl} \vec{A}$ would also be possible, but this would lead to all-zero field components when imposing the boundary conditions.)

Solution (271) is called a TEM wave because it only consists of transversal field components. An electrostatic field develops between the two separate conductors, resulting in a cutoff frequency $\omega_c = 0$. This cutoff frequency allows the flow of a direct current, which requires at least two separate conductors.

3.3 Attenuation of modes in waveguides

3.3.1 The power-loss method

In the previous sections the walls of the waveguides were assumed to be Perfectly Electric Conducting (PEC). Under this condition and above their cutoff frequency the waves propagate without attenuation, and the phase constant is a real number $k_z = \beta$.

In practical applications the walls only have a finite conductivity, leading to a complex propagation constant $k_z = \beta - j\alpha$ for each wave in a waveguide, and to attenuated waves even above cutoff. As an exact calculation of the losses usually is very costly, we introduce some approximations.

A common approach is the power-loss method. Here we assume that if the conductivity of the walls is sufficiently high, the electromagnetic fields can be approximated by the fields of a guide with PEC boundaries. From these fields an approximation of the power of the wall losses per unit length can be derived.

The power loss P' per unit length of a wave propagating in the $+z$-direction is given by

$$P' = -\frac{dN}{dz} , \tag{272}$$

where N is the power flowing through the cross section A of the guide. In the presence of losses, the field vectors decrease in the z-direction like

$$|\underline{E}, \underline{H}| \sim e^{-\alpha z} . \tag{273}$$

As the power through the cross section is proportional to the product of the electric and the magnetic field, we obtain

$$N(z) = N_0 \, e^{-2\alpha z} . \tag{274}$$

From Eq. (272) we get

$$P' = -\frac{dN}{dz} = 2\alpha N \tag{275}$$

and finally

$$\alpha = \frac{1}{2}\frac{P'}{N} . \tag{276}$$

In the power-loss method, the transported power N as well as the power loss P' per unit length are derived from the fields of the loss-free guide. For N we integrate the Poynting vector over the cross section A:

$$N = \frac{1}{2}\mathrm{Re}\left\{ \int_A (\underline{\vec{E}} \times \underline{\vec{H}}^*) \cdot \mathrm{d}\vec{A} \right\} . \tag{277}$$

For the derivation of the power loss in the wall, we consider the surface current in the loss-free case:

$$\vec{J}_s = \vec{n} \times \vec{H} \quad \Rightarrow \quad |\vec{J}_s| = H_{tan} \tag{278}$$

according to Eq. (80), where H_{tan} is the tangential magnetic field at the wall. With a finite conductivity the currents in the wall (normal direction x) decrease like

$$J(x) \sim e^{-(1-i)\frac{x}{\delta}} \tag{279}$$

(cf. formulas (128)–(130) for a good conductor), where

$$\delta = \sqrt{\frac{2}{\omega\kappa\mu}} \tag{280}$$

is the skin depth of the waveguide wall. The assumption

$$\int_0^\infty J(x)dx \approx H_{tan} \tag{281}$$

finally leads to

$$P' = \ldots = \frac{1}{2} R_m \int_{\partial A} |\underline{H}_{tan}|^2 \, ds \qquad \text{with} \quad R_m = \frac{1}{\delta\kappa} = \sqrt{\frac{\mu\omega}{2\kappa}} \,, \tag{282}$$

where the integration has to be performed along the boundary ∂A of the cross section of the guide.

3.3.2 Applicability of the power-loss method

The power-loss method is only applicable if the wall currents themselves do not excite other wave types in the guide. This may happen, however, if new field components arise by the transition from infinite to finite conductivity.

Unfortunately this is the case for all modes in rectangular waveguides except for the TE$_{m0}$- and TE$_{0n}$-modes (as their TM counterparts do not exist). In circular waveguides, however, the method is applicable to all TE and TM modes.

Criteria for the applicability of the power-loss method are summarized in Fig. 29.

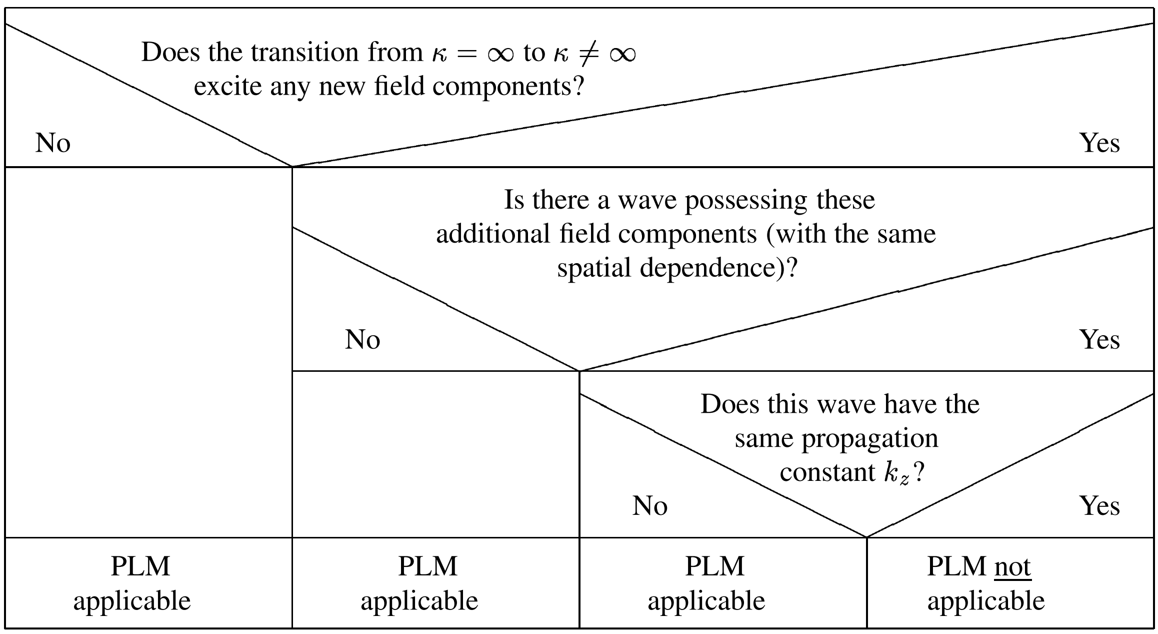

Fig. 29: Criteria for the applicability of the Power-Loss Method (PLM)

4 RESONANT CAVITIES

4.1 Solution of Maxwell's equations in cavities

In this section we consider the electromagnetic field inside resonant cavities, and again start with the assumption of PEC walls. Analogously to the previous derivation, an approach with a complex vector potential $\underline{\vec{A}}$ can be chosen, with

$$\underline{\vec{E}} = \mathrm{curl}\,\underline{\vec{A}} \quad \text{or} \quad \underline{\vec{H}} = \mathrm{curl}\,\underline{\vec{A}}\ .$$

From Maxwell's equations we obtain the wave equation (190)

$$\Delta\underline{\vec{A}} + k^2\underline{\vec{A}} = 0\ .$$

An analytical solution of this equation is only possible for a few cases, particularly if the surfaces bounding the cavity are coordinate surfaces (allowing the separation of variables). Among such cavities one should point out rectangular resonators (in a Cartesian coordinate system), and cylindrical and coaxial resonators (in a cylindrical coordinate system). In practice, cavities shaped close to cylindrical and rectangular resonators are the most frequent design.

A simple approach for these types of cavities is to interpret them as waveguides with both ends closed. This leads to a classification of the eigenmodes of the cavity, which is similar to the waveguide one:

- TM modes (E modes), characterized by a non-zero longitudinal component of the electric field and by purely transverse magnetic fields,
- TE modes (H modes), characterized by a non-zero longitudinal component of the magnetic field and by purely transverse electric fields.

Note that one distinguished coordinate direction ('longitudinal') is introduced by this classification. In the field of accelerator cavities this is usually the beam direction.

4.2 Rectangular cavities with perfectly conducting walls

If we superpose two waves in a loss-free waveguide, which have identical mode patterns $\underline{\vec{E}}(x, y)$ but propagate in opposite directions ($\pm z$), we obtain a standing wave:

$$\underline{\vec{E}}(x,y) \cdot (e^{-ik_z z} + e^{+ik_z z}) = \underline{\vec{E}}(x,y) \cdot 2\,\cos(k_z z)\ . \tag{283}$$

For TE waves, with $\underline{\vec{E}}(x, y)$ having only transversal components, the electric field vanishes for all

$$k_z z = \pi/2 + n\pi \qquad \Rightarrow \quad \cos(k_z z) = 0 \qquad (n \in Z)\ . \tag{284}$$

(A similar condition can also be found for the transversal components of TM waves.) Thus, if we insert perfectly conducting walls at a distance

$$\Delta z_p = L = \frac{p\pi}{k_z} \qquad (p \in N)\ , \tag{285}$$

the fields do not change and build the electromagnetic fields of a corresponding mode in a cavity with length L (see Fig. 30).

As the propagating constant k_z of the waveguide modes depends on the frequency of the fields, the condition (285) cannot be fulfilled at arbitrary frequencies if the geometry of the cavity (the length Δz) is kept fixed. Instead of the dispersion relation (224) of the waveguide (with a continuous relation between k_z and ω), we obtain the equation

$$\begin{aligned} k_x^2 + k_y^2 + k_z^2 &= k^2 = \left(\frac{\omega}{c}\right)^2 \\ \Rightarrow \quad \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{\Delta z}\right)^2 &= \left(\frac{\omega}{c}\right)^2\ , \end{aligned} \tag{286}$$

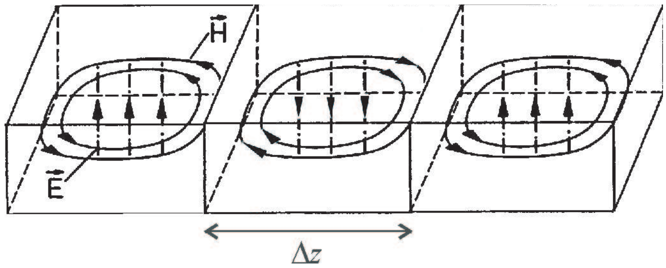

Fig. 30: Derivation of cavity modes: the fields of a standing wave (here, $H_{z,10}$ mode) are not changed if PEC walls are inserted

which defines a discrete (but infinite) set of eigensolutions in a cavity, including a discrete set of eigen-frequencies or resonance frequencies

$$f_{mnp} = \frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{L}\right)^2} \, . \tag{287}$$

For the field components in the cavity we obtain the following expressions, first for TM modes:

$$\underline{E}_x = \underline{C}\,\frac{mp\pi^2}{aL}\frac{1}{k}\frac{1}{\kappa_e}\cos\left(\frac{m\,\pi}{a}\,x\right)\,\sin\left(\frac{n\,\pi}{b}\,y\right)\,\sin\left(\frac{p\,\pi}{L}\,z\right) \, , \tag{288}$$

$$\underline{E}_y = \underline{C}\,\frac{np\pi^2}{bL}\frac{1}{k}\frac{1}{\kappa_e}\sin\left(\frac{m\,\pi}{a}\,x\right)\,\cos\left(\frac{n\,\pi}{b}\,y\right)\,\sin\left(\frac{p\,\pi}{L}\,z\right) \, , \tag{289}$$

$$\underline{E}_z = \underline{C}\,\frac{\kappa_e}{k}\sin\left(\frac{m\,\pi}{a}\,x\right)\,\sin\left(\frac{n\,\pi}{b}\,y\right)\,\cos\left(\frac{p\,\pi}{L}\,z\right) \, , \tag{290}$$

$$\underline{H}_x = -\underline{C}\,\frac{1}{\kappa_e}\frac{n\pi}{b}\sin\left(\frac{m\,\pi}{a}\,x\right)\,\cos\left(\frac{n\,\pi}{b}\,y\right)\,\cos\left(\frac{p\,\pi}{L}\,z\right) \, , \tag{291}$$

$$\underline{H}_y = \underline{C}\,\frac{1}{\kappa_e}\frac{m\pi}{a}\cos\left(\frac{m\,\pi}{a}\,x\right)\,\sin\left(\frac{n\,\pi}{b}\,y\right)\,\cos\left(\frac{p\,\pi}{L}\,z\right) \, , \tag{292}$$

$$\underline{H}_z = 0 \, . \tag{293}$$

And for TE cavity modes:

$$\underline{E}_x = -\underline{C}\,\frac{1}{\kappa_h}\frac{n\pi}{b}\cos\left(\frac{m\,\pi}{a}\,x\right)\,\sin\left(\frac{n\,\pi}{b}\,y\right)\,\sin\left(\frac{p\,\pi}{L}\,z\right) \, , \tag{294}$$

$$\underline{E}_y = \underline{C}\,\frac{1}{\kappa_h}\frac{m\pi}{a}\sin\left(\frac{m\,\pi}{a}\,x\right)\,\cos\left(\frac{n\,\pi}{b}\,y\right)\,\sin\left(\frac{p\,\pi}{L}\,z\right) \, , \tag{295}$$

$$\underline{E}_z = 0 \, , \tag{296}$$

$$\underline{H}_x = -\underline{C}\,\frac{mp\pi^2}{aL}\frac{1}{\kappa_h}\frac{1}{k}\sin\left(\frac{m\,\pi}{a}\,x\right)\,\cos\left(\frac{n\,\pi}{b}\,y\right)\,\cos\left(\frac{p\,\pi}{L}\,z\right) \, , \tag{297}$$

$$\underline{H}_y = -\underline{C}\,\frac{np\pi^2}{bL}\frac{1}{\kappa_h}\frac{1}{k}\cos\left(\frac{m\,\pi}{a}\,x\right)\,\sin\left(\frac{n\,\pi}{b}\,y\right)\,\cos\left(\frac{p\,\pi}{L}\,z\right) \, , \tag{298}$$

$$\underline{H}_z = \underline{C}\,\frac{k}{\kappa_h}\cos\left(\frac{m\,\pi}{a}\,x\right)\,\cos\left(\frac{n\,\pi}{b}\,y\right)\,\sin\left(\frac{p\,\pi}{L}\,z\right) \, , \tag{299}$$

with the abbreviations

$$\kappa_e = \kappa_h = \sqrt{(m\pi/a)^2 + (n\pi/b)^2} \, . \tag{300}$$

As we can see from these formulas, for TE modes $p \geq 1$ is required to obtain a non-zero field solution, whereas $m = 0$ or $n = 0$ is allowed (but, like for waveguides, *not* $m = n = 0$).

For the TM case, we must have $m, n \geq 1$ (as for waveguides). However, the case $p = 0$ is also possible and leads to a three-component field ($\underline{H}_x, \underline{H}_y, \underline{E}_z$), which cannot be derived as a superposition of waveguide modes, but corresponds to the TM$_{mn}$ waveguide mode at cutoff.

The cavity modes can be sorted by referring to their resonance frequency. When the resonator dimensions are such that $a \geq b \geq L$, the lowest resonance frequency is found to be

$$f_{110} = \frac{c}{2\pi}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} \tag{301}$$

with $m = n = 1$ and $p = 0$, corresponding to the TM$_{110}$ cavity mode. For a cavity with $a = 6$ cm and $b = 3$ cm the resonance frequency is $f_{110} = 5.59$ GHz.

The field distribution is illustrated in Fig. 31. We see that the electric fields are perpendicular to

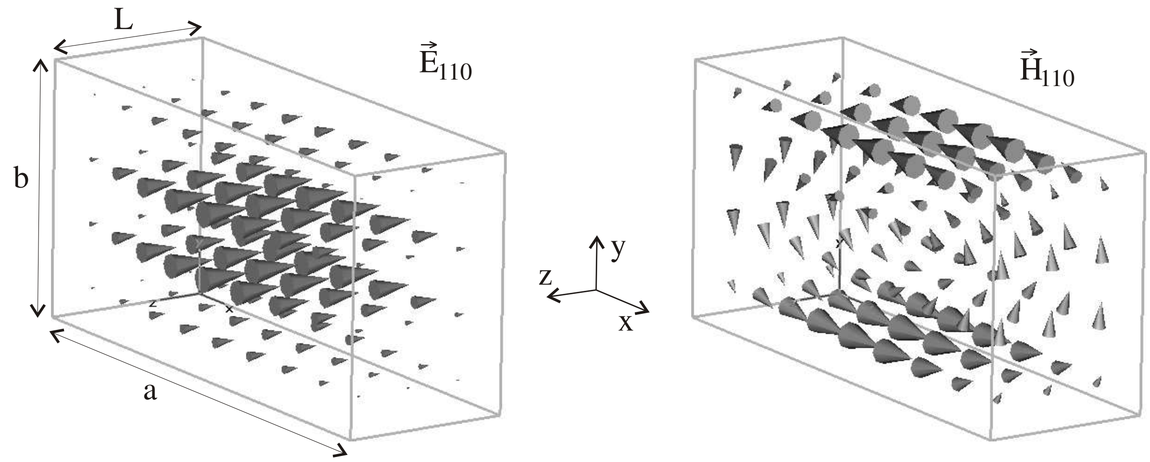

Fig. 31: Electric and magnetic field pattern of the $E_{z,110}$ mode in a rectangular cavity

the plate boundaries at $z = 0$ and $z = L$ and concentrate at the centre of the cavity so that the tangential $\vec{E}$ field vanishes at the boundaries $x = 0, a$ and $y = 0, b$. This field can also be considered as a dominant waveguide mode propagating in the y-direction and reflected at the walls $y = 0$ and $y = b$ to form a standing wave. If the labels of the coordinate axes y and z are interchanged, this mode may also be called a TE$_{101}$ mode. To have an unambiguous notation we have to refer to $E_{z,101}$.

4.3 Cylindrical cavities

A similar derivation can be performed for the fields in cylindrical cavities. Starting with the superposition of TE and TM waveguide modes forming a standing wave and fulfilling the boundary conditions at two electric walls, we again obtain discrete sets of cavity eigenmodes.

For the resonance frequencies in a cylindrical cavity with radius R and length L we get the TM solutions

$$f_{mnp}^{(\mathrm{TM})} = \frac{c}{2\pi}\sqrt{\left(\frac{j_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}\,, \tag{302}$$

where j_{mn} is the nth root of the Bessel function J_m. For non-zero fields $n \geq 1$ is required, whereas $m = 0$ and $p = 0$ are allowed.

The eigenfrequencies of the TE solutions are

$$f_{mnp}^{(\mathrm{TE})} = \frac{c}{2\pi}\sqrt{\left(\frac{j'_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}\,, \tag{303}$$

where now j'_{mn} is the nth root of J'_m, and $m, n, p \geq 1$ is required.

The fundamental TM mode in a cylindrical cavity is the TM_{010} mode, which corresponds to the waveguide mode TM_{01} at cutoff (see Fig. 32). Its resonance frequency is

$$f_{010}^{(\mathrm{TM})} = \frac{c\,2.405}{2\pi\,R}\,, \tag{304}$$

or $f_{010}^{(\mathrm{TM})} = 3.825$ GHz for a cavity with radius $R = 3$ cm.

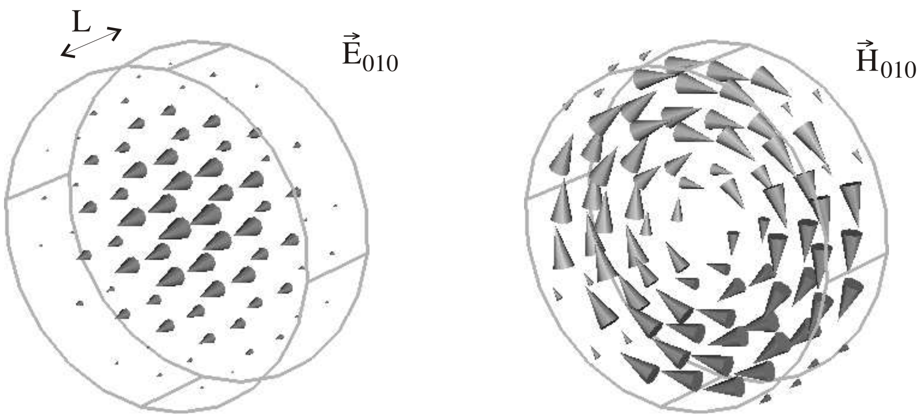

Fig. 32: Electric and magnetic field pattern of the $E_{z,010}$ mode in a cylindrical cavity

The lowest TE mode is the TE_{111} mode. Its resonance frequency is

$$f_{111}^{(\mathrm{TE})} = \frac{c}{2\pi}\sqrt{\left(\frac{1.841}{R}\right)^2 + \left(\frac{\pi}{L}\right)^2}\,, \tag{305}$$

or $f_{111}^{(\mathrm{TE})} = 8.046$ GHz for a cavity with radius $R = 3$ cm and length $L = 2$ cm.

For small L, the TM_{010} mode is the fundamental mode in a cylindrical cavity. For large L ($L > 2.03R$), however, we have $f_{111}^{(\mathrm{TE})} < f_{010}^{(\mathrm{TM})}$, and the TE_{111} mode is the fundamental. Because the frequency of this mode depends on the ratio R/L, it is possible to provide easy tuning by making the separation of the end faces adjustable.

Most cavities used in practice have a more complicated geometry, making analytical solutions more difficult or even not accessible at all. For such cavities, the eigensolutions (fields and resonance frequencies) can only be computed using numerical methods.

4.4 Losses in walls with finite conductivity, Q values

In the preceding section we neglected all sources of energy losses in a cavity, like non-perfectly electric conducting walls or conducting materials (e.g. a non-perfect vacuum) inside the cavity. Under this assumption we obtained a discrete set of eigensolutions with eigenfrequencies ω_i, corresponding to electric and magnetic fields oscillating at one specific frequency without any attenuation.

In the presence of losses, however, the cavity modes no longer have a sharp delta function singularity on the frequency axes (at their resonance frequency), but rather a narrow band of frequencies occurs around the eigenfrequency. In time domain with no excitations, this is equivalent to free attenuated oscillations with a time dependency such as

$$\vec{E}, \vec{H} \sim e^{i\omega t - \alpha t}\,. \tag{306}$$

A measure of the sharpness of response of the cavity to external excitation is the quality factor Q of a cavity mode, defined as the ratio of the time-averaged energy W stored in the cavity to the energy loss per cycle:

$$Q_0 = \omega \frac{W}{P_d} \, , \tag{307}$$

where P_d is the dissipated power in the cavity. From Eq. (306) and

$$W \sim e^{-2\alpha t} \quad \Rightarrow \quad P_d = -\dot{W} = 2\alpha W \tag{308}$$

we find

$$Q_0 = \frac{\omega}{2\alpha} \, . \tag{309}$$

To calculate the Q-value of a cavity mode due to wall losses, we again apply the power-loss approach (see formulas (272)–(282)): we approximate the fields in the lossy cavity by the solution of the loss-free case, and integrate the power loss of the wall currents over the surface of the cavity, and the stored energy over the volume V of the cavity:

$$W = \int_V w \, dV = \frac{1}{2} \int_V \left(\frac{\varepsilon}{2} |\underline{\vec{E}}|^2 + \frac{\mu}{2} |\underline{\vec{H}}|^2 \right) dV \, , \tag{310}$$

$$P_d = \int_{\partial V} P_d' dA = \frac{1}{2} \int_{\partial V} \sqrt{\frac{\omega\mu}{2\kappa}} \, |H_{tan}|^2 dA \, . \tag{311}$$

For rectangular and cylindrical cavities these integrals can easily be evaluated. The Q-values for some cavities with copper walls ($\kappa \approx 5e7$ S/m) are summarized in the following table:

Rectangular	a	b	L	Type	f	Q
	6 cm	3 cm	2 cm	TM_{110}	5.59 GHz	1.03e4
	6 cm	3 cm	2 cm	TE_{111}	9.35 GHz	1.18e4
	5 m	4 m	3 m	TM_{110}	69.3 MHz	6.60e4
Cylindrical		R	L			
		3 cm	2 cm	TM_{010}	3.83 GHz	1.15e4
		1 m	1 m	TM_{010}	0.11 GHz	8.10e4
		1 m	1 m	TE_{111}	0.17 GHz	9.22e4

The Q-values of cavities with a more complex geometrical shape can be evaluated from the results of a numerical field simulation.

Other sources of loss are the material filling of the cavity, surface irregularities of the cavity walls, and the coupling between other systems. They all contribute to the power dissipation P_d and therefore decrease Q_0.

4.5 The main electrodynamic characteristics of accelerator cavities

In charged particle accelerators two types of operation are common, namely Travelling Wave (TW) and Standing Wave (SW). In the TW case, the electrodynamic structure is an accelerating waveguide; in the SW case, an accelerating resonator in the form of a single cavity or a chain of coupled cavities is used [5].

The main characteristics of accelerating cavities are the resonant frequency, the quality factor (307), the cavity-generator coupling factor, and the effective shunt impedance.

In the absence of losses in the walls and the cavity-filling medium (or vacuum), the resonance frequency of any mode is a real valued quantity. For the simple form cavities, e.g. the 'pillbox' cavity (a circular cylindrical resonator), the frequency can be calculated analytically, as was shown above. In resonators of intricate geometry, these frequencies have to be determined by numerical methods or experimentally.

The cavity-generator coupling factor β_0 is the ratio of the unloaded quality factor Q_0 to the external quality factor Q_{ext} (a measure for the coupling of an excitation to the fields in the cavity):

$$\beta_0 = \frac{Q_0}{Q_{ext}} \, . \tag{312}$$

As a rule, the coupling factor of the beam-loaded cavity must be close to unity [5] (critical coupling), because then maximum power transmitted from the generator to the resonator. The power transmission factor is

$$k_t = 1 - |\Gamma_{in}|^2 = \frac{k_{t0}}{1 + a_1^2} \, , \tag{313}$$

where Γ_{in} is the reflection coefficient at the cavity input, k_{t0} is the transmission coefficient at the resonance frequency ω_0,

$$k_{t0} = \frac{4\beta_0}{(1+\beta_0)^2} \, , \tag{314}$$

and a_1 is the generalized detuning of the loaded resonator,

$$a_1 = 2\, Q_1 \, \frac{\Delta\omega}{\omega_0} \, . \tag{315}$$

In the last formula, $\Delta\omega = \omega - \omega_0$ is the frequency detuning, and Q_1 is the loaded quality factor given by

$$\frac{1}{Q_1} = \frac{1}{Q_0} + \frac{1}{Q_{ext}} \, . \tag{316}$$

The effective shunt impedance in the accelerating cavity $R_{sh,eff}$ relates the equivalent voltage U between two points in the cavity over a given path to the power P_d dissipated in the cavity walls:[2]

$$R_{sh,eff} = \frac{U^2}{2\, P_d} \, , \tag{317}$$

where

$$U = \left| \int\limits_{z_1}^{z_2} \underline{E}_z(z) \, \exp(ik_z z) dz \right| \, . \tag{318}$$

$\underline{E}_z(z)$ is the distribution of the longitudinal component of the electric field over the cavity axis, and z_1 and z_2 are the points on the cavity axis such that $z_2 - z_1 = L$ is the cavity length.

The effective shunt impedance, Eq. (317), takes into account the field variation in the cavity during the passage of accelerated particles. The transit time factor is given by

$$T = \frac{\left| \int_0^L \underline{E}_z(z) \, \exp(ik_z z) dz \right|}{\int_0^L |\underline{E}_z(z)| dz} \tag{319}$$

[2] Note that often the definition $R = U^2/P_d$ is used, and thus the shunt impedance in other papers may be two times higher.

with

$$R_{sh,eff} = R_{sh} T^2 \ , \tag{320}$$

where the shunt impedance of the accelerating cavity is

$$R_{sh} = \frac{U_0^2}{2\,P_d} = \frac{1}{2\,P_d} \left(\int_0^L |\underline{E}_z(z)|^2 dz \right)^2 \ . \tag{321}$$

The effective shunt impedance per unit length $r_{sh,eff}$ is given as follows:

$$r_{sh,eff} = \frac{1}{2\,P_d L} \left| \int_0^L \underline{E}_z(z) \, \exp(ik_z z) dz \right|^2 \ . \tag{322}$$

Using the expressions for shunt impedance and quality factor we may write the characteristic impedance as

$$\frac{R_{sh}}{Q} = \frac{1}{2\,\omega W} \left(\int_0^L |\underline{E}_z(z)|^2 dz \right)^2 \ . \tag{323}$$

or

$$\frac{R_{sh,eff}}{Q} = \frac{1}{2\,\omega W} \left| \int_0^L \underline{E}_z(z) \, \exp(ik_z z) dz \right|^2 \ . \tag{324}$$

The ratio R_{sh}/Q depends on the cavity geometry and does not depend on the cavity's material and quality of element matching. This ratio is sometimes called the characteristic impedance.

REFERENCES

[1] J.C. Maxwell, *A Treatise on Electricity and Magnetism* (Oxford University Press, London, 1873).

[2] K. Simony, *Theoretische Elektrotechnik* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1973).

[3] J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).

[4] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1965).

[5] N.P. Sobenin and B.V. Zverev, *Electrodynamic Characteristics of Accelerating Cavities* (MEPIUP, London and Moscow, 1999).