BASIC RF THEORY, WAVEGUIDES AND CAVITIES

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ABSTRACT
After a brief mathematical introduction, Maxwell's equations are discussed in their most general form as known today. Plane waves are considered in unbounded media, with the laws of reflection and refraction at a planar interface. A general theory of waveguides and cavities is then developed in detail. Periodically loaded waveguides are given as an example of accelerating structure. The parameters which characterize the interaction of resonant cavities with a particle beam are introduced, for both longitudinal and transverse motions.

1. MATHEMATICAL REPRESENTATION OF PHYSICAL VARIABLES AS A FUNCTION OF TIME

1.1 Sinusoidal variables (time-harmonic electromagnetic fields)

Phasors

Real sinusoidal variables

$$a(t) = a_m \cos(\omega t + \phi)$$

are represented as

$$\mathrm{Re}\left[a_m e^{j(\omega t+\phi)}\right] = \mathrm{Re}\left[A\ e^{j\omega t}\right] \tag{1.1}$$

where $A\ e^{j\omega t}$ (with $\omega \geq 0$) is called a *phasor* and $A = a_m\ e^{j\phi}$ is its complex amplitude.

When the differential equations determining the physical variable $a(t)$ are linear with real coefficients, any complex solution of the type $A\ e^{j\omega t}$ yields two real solutions which represent physical quantities:

$$\mathrm{Re}\left(A\ e^{j\omega t}\right) = |A| \cos(\omega t + \phi)$$

$$\mathrm{Im}\left(A\ e^{j\omega t}\right) = |A| \sin(\omega t + \phi)$$

Since these solutions merely differ by the time origin, it is sufficient to keep only the first one.

A (capital letter) is the complex amplitude of the real sinusoidal variable $a(t)$ written as a lower case letter.

$|A|$ is the amplitude a_m; $\arg(A) = \phi$ is the phase of $a(t)$.

In Eq. (1.1), the phasor $A\ e^{j\omega t}$ may be replaced by its complex conjugate; in fact both conventions are used in the literature. Here we will adhere to the $e^{j\omega t}$ convention because it is always used in RF engineering, in particular in all measuring instruments; the $e^{-i\omega t}$ convention is often used in theoretical physics. In order to translate results from one convention to the other, simply replace j by $(-i)$. The usefulness of phasors stems from the fact that they are eigenfunctions of the operator $\partial/\partial t$, with eigenvalue $j\omega$. The exponential $e^{j\omega t}$ can be factorized out of all linear differential equations; finally the equations are written using the complex amplitudes of all variables, with $\partial/\partial t$ replaced by $j\omega$. This formalism also applies to sinusoidal variables which are exponentially damped with time. In that case ω is complex:

$$j\omega = j\omega_1 - \alpha_1 \quad \text{where} \quad \omega_1, \alpha_1 \ \text{ are real positive quantities}. \tag{1.2}$$

Time average of the product of two sinusoidal variables

If $a_1(t) = \mathrm{Re}[A_1\, e^{j\omega t}]$, $a_2(t) = \mathrm{Re}[A_2\, e^{j\omega t}]$, we have

$$a_1(t)a_2(t) = \frac{1}{2}\left[A_1 e^{-\alpha_1 t + j\omega_1 t} + A_1^* e^{-\alpha_1 t - j\omega_1 t}\right] \cdot \frac{1}{2}\left[A_2 e^{-\alpha_1 t + j\omega_1 t} + A_2^* e^{-\alpha_1 t - j\omega_1 t}\right]$$

$$= \frac{1}{4}\left[A_1 A_2^* + A_1^* A_2 + A_1 A_2\ e^{2j\omega_1 t} + A_1^* A_2^*\ e^{-2j\omega_1 t}\right]\ e^{-2\alpha_1 t}$$

$$= \frac{1}{2}\left[\mathrm{Re}\left(A_1 A_2^*\right) + \mathrm{Re}\left(A_1 A_2\ e^{2j\omega_1 t}\right)\right]\ e^{-2\alpha_1 t} \tag{1.3}$$

In Eq. (1.3), the first term is slowly (or not at all) varying with time; the second term is oscillating at a frequency $2\omega_1$ and will be discarded. If the variables are damped, it is convenient to consider that the factor $e^{-\alpha_1 t}$ is included in their complex amplitudes, so that (1.3) reads

$$\overline{a_1(t)a_2(t)} = \mathrm{Re}\left(\frac{1}{2} A_1 A_2^*\right) = \mathrm{Re}\left(\frac{1}{2} A_1^* A_2\right) \tag{1.4}$$

Examples:

$$\overline{P} = \mathrm{Re}\left(\frac{1}{2} V I^*\right); \quad \frac{1}{2} V I^* \text{ is called the complex power} \tag{1.4a}$$

$$\overline{\upsilon^2(t)} = \frac{1}{2}|V|^2 \tag{1.4b}$$

Vector phasors

Real vectors which vary sinusoidally with time are represented by 3-component phasors:

$$\vec{a}(t) = \mathrm{Re}\left[\vec{A}\ e^{j\omega t}\right]$$

If the vector $\vec{A}$ is real or if it is proportional to a real vector, the physical vector $\vec{a}(t)$ is linearly polarized along the direction defined by $\vec{A}$. If the vector $\vec{A}$ is not proportional to a real vector (we then say that $\vec{A}$ is complex), it is possible to find a real vector $\vec{n}$ of unit length such that $\vec{n} \cdot \vec{A} = 0$. The physical vector $\vec{a}(t)$ then always lies in a plane perpendicular to $\vec{n}$ where its extremity describes an ellipse in the course of time: $\vec{a}(t)$ is said to be elliptically polarized.

Scalar (or dot) product

From (1.3) we have

$$\vec{a}(t) \cdot \vec{b}(t) = \frac{1}{2}\left[\mathrm{Re}\left(\vec{A} \cdot \vec{B}^*\right) + \mathrm{Re}\left(\vec{A} \cdot \vec{B}\ e^{2j\omega_1 t}\right)\right]\ e^{-2\alpha_1 t}$$

The slowly (or not at all) varying term involves $\vec{A} \cdot \vec{B}^*$, whereas the oscillating term involves $\vec{A} \cdot \vec{B}$. In particular, if $\vec{A} \cdot \vec{B} = 0$ the scalar product $\vec{a}(t) \cdot \vec{b}(t)$ is slowly (or not at all) varying with time, but its value is not zero unless $\mathrm{Re}\left(\vec{A} \cdot \vec{B}^*\right) = \mathrm{Re}\left(\vec{A}^* \cdot \vec{B}\right) = 0$; this will be the case, for example, if one of the vector $\vec{A}$ or $\vec{B}$ is real, which means that one of the physical vectors $\vec{a}(t)$ or $\vec{b}(t)$ is linearly polarized.

$$\text{Time average of } \vec{a}(t) \cdot \vec{b}(t) = \mathrm{Re}\frac{1}{2}\left(A_x B_x^* + A_y B_y^* + A_z B_z^*\right) = \mathrm{Re}\frac{1}{2}\left(\vec{A} \cdot \vec{B}^*\right)$$

$$\text{Time average of } \vec{e}(t) \cdot \vec{e}(t) = \frac{1}{2}\left(|E_x|^2 + |E_y|^2 + |E_z|^2\right) = \frac{1}{2}|\vec{E}|^2$$

Example:

Time average of the electric energy density stored in a dielectric =

$$\overline{\frac{1}{2}\vec{e}\cdot\vec{d}} = \frac{1}{2}\mathrm{Re}\left(\frac{1}{2}\vec{E}\cdot\vec{D}^*\right) = \frac{1}{4}\mathrm{Re}(\varepsilon)\,|\vec{E}|^2$$

Vector (or cross) product

$$\left[\vec{E}\cdot\vec{H}^*\right]_z = E_x H_y^* - E_y H_x^* \tag{1.5}$$

$$\text{Time average of}\left[\vec{e}\times\vec{h}\right] = \mathrm{Re}\;\underbrace{\frac{1}{2}\left[\vec{E}\times\vec{H}^*\right]}_{\text{complex Poynting vector}} \tag{1.6}$$

1.2 General time variation

Fourier transforms

For a general time variation, we use Fourier transforms:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\ e^{j\omega t}\mathrm{d}\omega \qquad F(\omega) = \int_{-\infty}^{+\infty} f(t)\ e^{-j\omega t}\mathrm{d}t \tag{1.7}$$

This representation uses positive and negative frequencies, i.e. a two-sided frequency spectrum.

If $f(t)$ is real, $F(-\omega) = F^*(\omega)$ [1] and

$$f(t) = \mathrm{Re}\;\underbrace{\frac{1}{\pi}\int_0^{\infty} F(\omega)\ e^{j\omega t}\ d\omega}_{\substack{\text{Complex representation of the variable } f(t),\\ \text{as a superposition of phasors with positive}\\ \text{frequencies (one-sided frequency spectrum).}}} \tag{1.8}$$

Laplace transforms

Convergence condition: With ω real, in (1.7) $|f(t)|$ must decrease fast enough when $t \to \pm\infty$.

To get around this problem, we use unilateral Fourier transforms:

$$F_+(\omega) = \int_0^{\infty} f(t)\ e^{-j\omega t}\mathrm{d}t \qquad F_-(\omega) = \int_{-\infty}^{0} f(t)\ e^{-j\omega t}\mathrm{d}t \tag{1.9}$$

With $p = c + j\omega$,

$$F_+\left(\frac{p}{j}\right) = \int_0^{\infty} f(t)\ e^{-pt}\mathrm{d}t \qquad F_-\left(\frac{p}{j}\right) = \int_{-\infty}^{0} f(t)\ e^{-pt}\mathrm{d}t \tag{1.10}$$

F_+ may be continued analytically for $c \geq a$ where $a \leq 0$
F_- may be continued analytically for $c \leq b$ where $0 \leq b$.

Let $H(t)$ be Heaviside's unit step function:

$$H(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1/2 & \text{for } t = 0 \\ 1 & \text{for } t > 0 \end{cases}$$

$F_+(\omega)$ is the Fourier transform of the causal function $f(t)\,H(t)$; $F_+(p/j)$ is its Laplace transform.

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[1] This relation holds for ω real; for ω complex we would have $F(-\omega) = F^*(\omega^*)$.

Inversion formula for Laplace transform:

$$f(t)H(t)=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F_+\left(\frac{p}{j}\right)\,e^{pt}dp \qquad a\le c$$

$$f(t)H(-t)=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F_-\left(\frac{p}{j}\right)\,e^{pt}dp \qquad c\le b$$

$$f(t)=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F\left(\frac{p}{j}\right)\,e^{pt}dp \qquad a\le c\le b \tag{1.11}$$

Hilbert transforms

The Hilbert transform of a complex function $f(x)$ defined on the real axis is another complex function $g(y)$ also defined on the real axis by

$$\mathcal{H}[f(x)]=g(y)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(x)}{x-y}dx \tag{1.12}$$

where the integral is taken as a Cauchy principal value.

It can be shown that

$$\mathcal{H}[g(x)]=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{g(x)}{x-y}dx \;=\; -f(y) \tag{1.13}$$

Examples

1) Hilbert transform of $e^{j\omega t}$ with respect to t

$$\mathcal{H}_t\left[e^{j\omega t}\right]=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{e^{j\omega t}}{t-t'}dt=\frac{1}{\pi}e^{j\omega t'}\int_{-\infty}^{+\infty}\frac{e^{j\omega t}}{t}dt=\frac{j}{\pi}e^{j\omega t'}\int_{-\infty}^{+\infty}\frac{\sin(\omega t)}{t}dt=j\;\; sgn(\omega)\,e^{j\omega t'} \tag{1.14}$$

2) Hilbert transform of $e^{j\omega t}$ with respect to ω

$$\mathcal{H}_\omega\left[e^{j\omega t}\right]=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{e^{j\omega t}}{\omega-\omega'}d\omega=j\;\;\mathrm{sgn}(t)\,e^{j\omega' t} \tag{1.15}$$

Causal function in time: $f(t) = 0$ for $t < 0$.

Its Fourier transform then reads

$$F(\omega)=\int_0^\infty f(t)\;e^{-j\omega t}dt$$

With (1.15)

$$\mathcal{H}[F(\omega)]=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{\omega-\omega'}\int_0^\infty f(t)\;e^{-j\omega t}dt=\int_0^\infty dt\;\;f(t)\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{\omega-\omega'}e^{-j\omega t}=-j\int_0^\infty dt\;\;f(t)\;\;e^{-j\omega' t}$$

$$\mathcal{H}[F(\omega)]=-j\;\;F(\omega') \quad \text{hence} \quad \begin{array}{l}\mathcal{H}[\mathrm{Re}\,F(\omega)]=\mathrm{Im}\,F(\omega')\\ \mathcal{H}[\mathrm{Im}\,F(\omega)]=-\mathrm{Re}\,F(\omega')\end{array} \tag{1.16}$$

Causal function in frequency: $F(\omega) = 0$ for $\omega < 0$.

Its Fourier transform reads

$$f(t)=\frac{1}{2\pi}\int_0^\infty F(\omega)\ e^{j\omega t}d\omega$$

With (1.14)

$$\mathcal{H}[f(t)]=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dt}{t-t'}\frac{1}{2\pi}\int_0^\infty F(\omega)\ e^{j\omega t}d\omega=\frac{1}{2\pi}\int_0^\infty d\omega\ \ F(\omega)\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dt}{t-t'}e^{j\omega t}=\frac{j}{2\pi}\int_0^\infty d\omega\ \ F(\omega)\ e^{j\omega t'}$$

$$\mathcal{H}[f(t)]=j\ f(t')\quad \text{hence}\quad \begin{array}{l}\mathcal{H}[\mathrm{Re}\,f(t)]=-\mathrm{Im}\,f(t')\\ \mathcal{H}[\mathrm{Im}\,f(t)]=\mathrm{Re}\,f(t')\end{array} \tag{1.17}$$

Application: In (1.8) we write

$$\frac{1}{\pi}\int_0^\infty F(\omega)\ e^{j\omega t}d\omega=f(t)+j\,g(t)=A(t)\ e^{j\varphi(t)}=s(t)\quad \text{with}\ \ f,g\ \ \text{and}\ A,\ \varphi\ \text{real} \tag{1.18}$$

The complex function $s(t)$ is of the type $f(t)$ in (1.17); therefore

$$\mathcal{H}[f(t)]=-g(t')\qquad \text{i.e.}\qquad g(t')=-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(t)}{t-t'}dt \tag{1.19}$$

To the real function $f(t)$ we can associate a complex signal $s(t)$ whose real part is $f(t)$, and whose imaginary part is obtained as $-\mathcal{H}\ [f(t)]$.

The modulus and argument of $s(t)$ allow a natural definition of the instantaneous amplitude, phase and frequency of the real function $f(t)$ as [1, 2, 3, 4]

$$A(t)=|s(t)|\qquad \phi(t)=\arg\left[s(t)\right]\qquad \Omega(t)=\frac{d\phi(t)}{dt} \tag{1.20}$$

Moreover

$$\int_{-\infty}^{+\infty}A^2(t)dt=\int_{-\infty}^{+\infty}dt\frac{1}{\pi}\int_0^\infty F(\omega)\ e^{j\omega t}d\omega\frac{1}{\pi}\int_0^\infty F^*(\omega')\ e^{-j\omega' t}d\omega'$$

$$=\frac{2}{\pi}\int_0^\infty d\omega\int_0^\infty d\omega' F(\omega)F^*(\omega')\ \ \delta(\omega-\omega')=\frac{2}{\pi}\int_0^\infty d\omega|F(\omega)|^2=\frac{1}{\pi}\int_{-\infty}^{+\infty}d\omega|F(\omega)|^2=2\int_{-\infty}^{+\infty}f^2(t)dt$$

from Parseval's formula applied to (1.7).

Therefore, since $A^2(t)=f^2(t)+g^2(t)$,

$$\int_{-\infty}^{+\infty}f^2(t)dt=\int_{-\infty}^{+\infty}g^2(t)dt=\frac{1}{2}\int_{-\infty}^{+\infty}A^2(t)dt \tag{1.21}$$

This relation is similar to (1.4b).

1.3 Useful vector identities

We will make repeated use of the following vector identities:

$$\vec{a}\cdot\left[\vec{b}\times\vec{c}\right]=\vec{b}\cdot\left[\vec{c}\times\vec{a}\right]=\vec{c}\cdot\left[\vec{a}\times\vec{b}\right]$$

$$\left[\vec{a}\times\left[\vec{b}\times\vec{c}\right]\right]=(\vec{a}\cdot\vec{c})\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

$$\left[\vec{a}\times\vec{b}\right]\cdot\left[\vec{c}\times\vec{d}\right]=(\vec{a}\cdot\vec{c})\left(\vec{b}\cdot\vec{d}\right)-\left(\vec{a}\cdot\vec{d}\right)\left(\vec{b}\cdot\vec{c}\right)$$

$$\operatorname{div}(U\vec{a}) = \vec{a}\cdot\operatorname{grad} U + U \operatorname{div}\vec{a} \qquad \operatorname{div}(U \operatorname{grad} V) = \operatorname{grad} U\cdot\operatorname{grad} V + U\,\Delta V$$

$$\operatorname{curl}(U\vec{a}) = [\operatorname{grad} U\times\vec{a}] + U \operatorname{curl}\vec{a} \qquad \operatorname{curl}\operatorname{grad} V = 0$$

$$\operatorname{div}\left[\vec{a}\times\vec{b}\right] = \vec{b}\cdot\operatorname{curl}\vec{a} - \vec{a}\cdot\operatorname{curl}\vec{b} \qquad \operatorname{div}\operatorname{curl}\vec{a} = 0$$

2. MAXWELL'S EQUATIONS AND SOME APPLICATIONS

2.1 Maxwell's equations

Let

$\vec{J}$ = free electric current density

ρ = free electric charge density

$\vec{J}_m$ = free magnetic current density

ρ_m = free magnetic charge density

Although magnetic currents and charges do not exist in nature, they are useful mathematical tools as equivalent sources for electromagnetic fields ([5], p. 46 and 52).

$$\operatorname{curl}\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \qquad \operatorname{div}\vec{D} = \rho$$

$$\operatorname{curl}\vec{E} = -\vec{J}_m - \frac{\partial\vec{B}}{\partial t} \qquad \operatorname{div}\vec{B} = \rho_m \tag{2.1}$$

Since div curl = 0,

$$\operatorname{div}\vec{J} + \frac{\partial\rho}{\partial t} = 0 \qquad \operatorname{div}\vec{J}_m + \frac{\partial\rho_m}{\partial t} = 0 \tag{2.2}$$

Equations (2.2) express the conservation of electric and magnetic charges.

Constitutive relations

These relations link the inductions $\vec{D}$, $\vec{B}$ to the fields $\vec{E}$, $\vec{H}$; they describe the electromagnetic properties of a medium. In general

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$
$$\vec{B} = \mu_0\vec{H} + \mu_0\vec{M} \tag{2.3}$$

where

$\vec{P}$ = electric polarization = electric dipole moment density

$\mu_0\vec{M}$ = magnetic polarization = magnetic dipole moment density

$\vec{M}$ = magnetisation (equivalent electric current = curl $\vec{M}$)

The definition (2.3) introduces an asymmetry between $\vec{P}$ and $\vec{M}$. This is an historical definition ([6], p. 12) and the most common one in the literature; but some authors do not write a factor μ_o in front of $\vec{M}$.

At a given time, $\vec{P}$ and $\vec{M}$ depend on the previous history of the sample. For linear media, by superposition $\vec{P}(t)$ can be written as a convolution

$$\vec{P}(t) = \int_{-\infty}^{t} p(t-t')\ \vec{E}\ (t')dt' = \int_{-\infty}^{+\infty} p(t-t')\ \vec{E}\ (t')dt' \tag{2.4}$$

where

$$p(t) = 0 \quad \text{for } t < 0,$$

which is the definition of a causal function in time.

In frequency domain with ω real, (2.4) becomes (here the Fourier transforms are represented by the same symbols as the time variables)

$$\vec{P}(\omega) = p(\omega)\ \vec{E}\ (\omega) \quad \text{with} \quad p(-\omega) = p^*(\omega)$$

so that

$$\vec{D}(\omega) = \varepsilon(\omega)\ \vec{E}\ (\omega) \quad \text{where} \quad \varepsilon(\omega) = \varepsilon_0 + p(\omega) \tag{2.5}$$

The relation between $\vec{D}$ and $\vec{E}$ is simple only in frequency domain. In general, $\varepsilon(\omega)$ is complex:

$$\varepsilon(\omega) = \varepsilon'(\omega) - j\varepsilon''(\omega) \quad \text{with} \quad \varepsilon(-\omega) = \varepsilon^*(\omega): \quad \varepsilon' \text{ even, } \varepsilon'' \text{ odd function of } \omega \tag{2.6}$$

Since $p(t)$ is a causal function, from (1.13) Re $p(\omega)$ = Re $[\varepsilon(\omega) - \varepsilon_o)]$ and Im $p(\omega)$ = Im $[\varepsilon(\omega)]$ are Hilbert transforms of each other:

$$-\varepsilon''(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\varepsilon'(\omega') - \varepsilon_0}{\omega' - \omega} d\omega' \qquad \varepsilon'(\omega) - \varepsilon_0 = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\varepsilon''(\omega')}{\omega' - \omega} d\omega' \tag{2.7}$$

Since ε' is an even and ε" is an odd function of ω, Eq. (2.7) transform in the Kramers-Kronig relations:

$$\varepsilon''(\omega) = -\frac{2\omega}{\pi}\int_{0}^{\infty} \frac{\varepsilon'(\omega') - \varepsilon_0}{\omega'^2 - \omega^2} d\omega' \qquad \varepsilon'(\omega) - \varepsilon_0 = \frac{2}{\pi}\int_{0}^{\infty} \frac{\omega'\varepsilon''(\omega')}{\omega'^2 - \omega^2} d\omega' \tag{2.8}$$

$\varepsilon''(\omega) \equiv 0$ would imply $\varepsilon'(\omega) \equiv \varepsilon_o$. For any dielectric medium, there are frequencies where $\varepsilon''(\omega) \neq 0$, which corresponds to a time lag between the polarization $\vec{P}(\omega)$ and the applied electric field $\vec{E}(\omega)$, i.e. hysteresis. We will show later (Eq. 2.38) that $\omega\, \varepsilon''(\omega) \geq 0$ in all cases whereas $[\varepsilon'(\omega) - \varepsilon_o]$ may take any sign. The loss tangent of the dielectric is defined as

$$\tan\ \delta = \frac{\varepsilon''}{\varepsilon'} \tag{2.9}$$

A typical behaviour of $\varepsilon(\omega)$ near a resonance is shown in Fig.2.1.

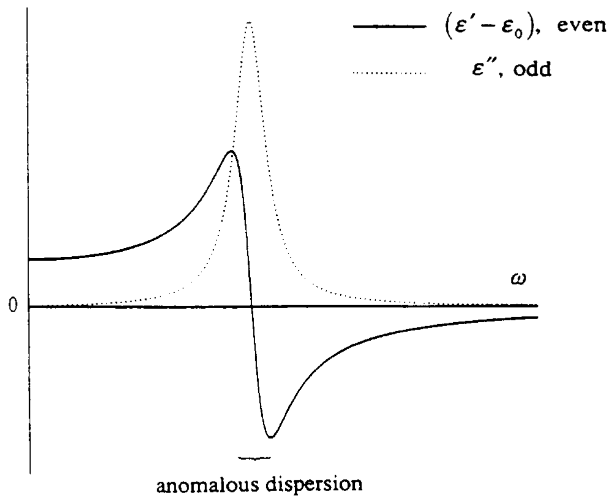


Fig.2.1 Absorption and dispersion in the neighbourhood of a resonance frequency

A simplified theory of dielectrics ([7], p. 32-7; [8], p. 19) yields

$$\frac{\varepsilon(\omega)-\varepsilon_0}{\varepsilon(\omega)+2\varepsilon_0}=\sum_n \frac{F_n}{\omega_n^2-\omega^2+j\alpha_n\omega} \qquad \text{with} \qquad \alpha_n << \omega_n$$

Similarly, we write
$$\vec{B}(\omega)=\mu(\omega)\vec{H}(\omega) \tag{2.10}$$

where
$$\mu(\omega)=\mu'(\omega)-j\mu''(\omega) \tag{2.11}$$

The variation of μ with frequency is more complicated than for ε. Nevertheless, the resonant type behaviour shown in Fig. 2.1 also applies for the effective μ_+ of a right-hand (+) circularly polarized wave in a ferrite ([8], p. 298).

Up to now we have implicitly assumed that the medium is isotropic, so that ε and μ are scalar quantities. For anisotropic materials, ε and μ are tensors of second rank, i.e. matrices (of order 3 in 3 dimensions). If ε, μ are symmetrical tensors, the material is reciprocal; if ε, μ are asymmetrical tensors, the material is non-reciprocal (like ferrites) ([9], p. 409). Loosely speaking, reciprocity means the possibility of interchanging source and detector (or generator and receiver) without affecting the results of measurements. In particular, in a reciprocal medium the propagation constant of a wave is the same for the two opposite directions of propagation.

The condition for reciprocity thus reads

$$\varepsilon=\tilde{\varepsilon} \qquad \mu=\tilde{\mu} \tag{2.12}$$

where a tilde represents the transposed matrix. It can be shown ([9], p. 53) that the condition for a medium to be lossless reads

$$\varepsilon=\tilde{\varepsilon}^*=\varepsilon^+ \qquad \mu=\tilde{\mu}^*=\mu^+ \tag{2.13}$$

where the + superscript represents the hermitian conjugate matrix.

In nonlinear materials (like ferromagnetic materials), ε and μ depend on the field strength.

For chiral media, the constitutive relations take the general form

$$\begin{aligned}\vec{D}&=\varepsilon\vec{E}+\xi\vec{H}\\ \vec{B}&=\eta\vec{E}+\mu\vec{H}\end{aligned} \tag{2.14}$$

where ξ and η have the same dimension as $\sqrt{\varepsilon_0\ \mu_0}$. This is the only form of constitutive relations which is Lorentz covariant ([9], p. 7). In the most general case ε, μ, ξ, η are second-rank tensors and the medium is called bianisotropic (bi because the constitutive relations involve both $\vec{E}$ and $\vec{H}$). It can be shown ([9], p.409) that the conditions (2.12) for reciprocity are supplemented by

$$\xi=-\tilde{\eta} \tag{2.15}$$

whereas the conditions (2.13) for a medium to be lossless must be supplemented by ([9], p. 53)

$$\xi=\tilde{\eta}^*=\eta^+ \tag{2.16}$$

These relations show that ξ or η cannot vanish without the other one also vanishing.

Example: Media which rotate the polarization plane of linearly polarized light. This is called natural optical activity; it does not require an external magnetic field as the Faraday rotation (which is nonreciprocal) ([8], p. 297).

In isotropic media with natural optical activity ([10], p. 19)

$$\xi = -j\sqrt{\varepsilon\mu}\,\frac{ka}{1-k^2a^2} \qquad \eta = j\sqrt{\varepsilon\mu}\,\frac{ka}{1-k^2a^2}$$

where $k^2 = \omega^2\varepsilon\mu$, and a is a length smaller than atomic dimensions; for quartz, $a \approx 0.01$ Å as deduced from ([13], p. 6-248).

Research is now going on to develop chiral materials for microwave frequencies.

Conduction current

Ohm's law states that

$$\vec{J} = \sigma\vec{E} \tag{2.17}$$

where σ is the electric conductivity of the medium. Strictly speaking, this relation is again only valid in frequency domain:

$$\sigma = \frac{\sigma_0}{1+j\omega\tau} \quad \text{with} \quad \sigma_0 = \varepsilon_0\omega_p^2\tau\ , \qquad \omega_p^2 = \frac{Ne^2}{\varepsilon_0\ m_e} \tag{2.18}$$

where N is the number of free electrons per unit volume, ω_p is their plasma frequency, and τ is the mean free time between collisions of the free electrons with the ion lattice ([7], p. 32-11; [11], p. 121; [12], p. 287). For copper at room temperature, $\tau = 2.4 \times 10^{-14}$ s; therefore, the approximation $\sigma = \sigma_o$ is valid up to frequencies such that $\omega\tau = 1$, which corresponds to $f = 6600$ GHz or $\lambda = 45$ µm. In the microwave range, we will thus consider that σ is independent of frequency.

Remark Due to the Hall effect ([13], p. 9-27), in the presence of a magnetic field the current density is such that

$$\vec{J} = \sigma\left(\vec{E} + R\left[\vec{J}\times\vec{B}\right]\right)$$

where R is called the Hall coefficient. From this relation it is easy to derive

$$\left\{1+\left(\sigma R\vec{B}\right)^2\right\}\vec{J} = \sigma\left\{\vec{E} + \sigma R\left[\vec{E}\times\vec{B}\right] + (\sigma R)^2\left(\vec{E}\cdot\vec{B}\right)\vec{B}\right\} \tag{2.19}$$

For copper at room temperature, $\sigma = 5.8 \times 10^7\ \Omega^{-1}\ \text{m}^{-1}$ and $R = -5.5 \times 10^{-11}\ \text{m}^3\ C^{-1}$ ([13], p. 9-39) hence $\sigma R = -3.2 \times 10^{-3}\ (\text{Tesla})^{-1}$. The relative correction introduced by (2.19) with respect to Ohm's law (2.17) is of the order σRB, that is -0.032 for $B = 10$ Tesla = 100 kG; this is negligible for all practical magnetic fields.

Total current

$$\text{Conduction current + displacement current} = \sigma\vec{E} + j\omega\vec{D} = (\sigma + j\omega\varepsilon)\vec{E}$$

When $\omega \neq 0$, the effect of a conduction current can be represented by the complex permittivity

$$\varepsilon + \frac{\sigma}{j\omega} = \varepsilon' - j\left(\varepsilon'' + \frac{\sigma}{\omega}\right) = \varepsilon' - j\,\frac{\sigma + \omega\varepsilon''}{\omega} \tag{2.20}$$

At a given frequency, the dielectric losses (ε'') cannot be distinguished from electric conductivity; in fact (2.20) shows that the electric conductivity may be considered as part of the complex permittivity.

Decay of electric charge inside a conductor

In frequency domain, (2.2) reads

$$\operatorname{div} \vec{J} + j\omega\rho = 0$$

With (2.17) and div $\vec{D} = \rho$ it becomes, if σ/ε is uniform in space:

$$\frac{\sigma}{\varepsilon}\rho + j\omega\rho = 0 \qquad \text{hence} \qquad \sigma + j\omega\varepsilon = 0$$

Using (2.18) we obtain

$$\sigma_0 + (1 + j\omega\tau) j\omega \ \varepsilon(\omega) = 0 \tag{2.21}$$

This equation determines the eigenfrequencies ω of the conductor. If we look for the eigenfrequencies which lie outside the absorption bands (these are due to bound electrons), we may take $\varepsilon(\omega) \approx \varepsilon_0$ and (2.21) becomes

$$(j\omega\tau)^2 + j\omega\tau + \frac{\sigma_0}{\varepsilon_0}\tau = 0 \ .$$

Therefore

$$j\omega\tau = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_0}{\varepsilon_0}\tau}$$

which means that

$$\rho = \rho_0 \ e^{-\frac{t}{\tau}\left[\frac{1}{2} \mp \sqrt{\frac{1}{4} - \frac{\sigma_0}{\varepsilon_0}\tau}\right]} \tag{2.22}$$

If $\frac{\sigma_0}{\varepsilon_0}\tau < \frac{1}{4}$, the charge density in the conductor decays as $e^{-t\frac{\sigma_0}{\varepsilon_0}\left[\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\sigma_0}{\varepsilon_0}\tau}\right]^{-1}}$

Even for distilled water, the time constant ε_o/σ_o is as short as 10^{-6} s.

If $\frac{\sigma_0}{\varepsilon_0}\tau = (\omega_p\tau)^2 > \frac{1}{4}$, the charge density decays as $e^{-\frac{t}{\tau}\left[\frac{1}{2} \mp j\sqrt{(\omega_p\tau)^2 - \frac{1}{4}}\right]}$, i.e. with a damping time 2τ ([12], p. 329); this is the case for all metals.

Poynting vector $\left[\vec{E} \times \vec{H}\right]$ in time domain.

From Maxwell's equations we can write

$$\underbrace{-\operatorname{div}\left[\vec{E} \times \vec{H}\right]}_{\text{inward flux of Poynting vector per unit volume}} = \vec{E} \cdot \operatorname{curl} \ \vec{H} - \vec{H} \cdot \operatorname{curl} \vec{E}$$

$$= \underbrace{\left(\vec{E} \cdot \vec{J} + \vec{H} \cdot \vec{J}_m\right)}_{\text{power released per unit volume}} + \underbrace{\left(\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}\right)}_{\frac{\partial W}{\partial t} \ \text{where} \ W \ = \ \text{energy density}} \tag{2.23}$$

We see that

$$\delta W = \vec{E} \cdot \delta\vec{D} + \vec{H} \cdot \delta\vec{B} \tag{2.24}$$

which is the correct expression for the work done in all cases, in particular when there is hysteresis, and part or all of the work is transformed into heat.

For *linear, nondispersive and lossless media,* where ε and μ are independent of field strength, of frequency, and satisfy (2.13), we have when $\xi = \eta = 0$:

$$W = \frac{1}{2}\vec{E} \cdot \vec{D} + \frac{1}{2}\vec{H} \cdot \vec{B} \tag{2.25}$$

For isotropic media (ε,μ scalar and real) this reduces to

$$W = \frac{1}{2}\varepsilon\vec{E}^2 + \frac{1}{2}\mu\vec{H}^2 \tag{2.26}$$

Complex Poynting vector and stored energy for time-harmonic electromagnetic fields

In the following, we consider a complex frequency

$$\omega = \omega_1 + j\alpha_1 \qquad j\omega = -\alpha_1 + j\omega_1 \qquad \text{with} \quad \alpha_1 << \omega_1 \tag{2.27}$$

Then, to first order in α_1 we have

$$\varepsilon(\omega) = \varepsilon(\omega_1) + j\alpha_1 \frac{d\varepsilon}{d\omega}(\omega_1) \; + \ldots \tag{2.28}$$

and similar expressions for μ, ξ, η (which are supposed to be second rank tensors).

The relation

$$-\text{div}\left[\vec{E} \times \vec{H}^*\right] = \vec{E} \cdot \text{curl}\,\vec{H}^* - \vec{H}^* \cdot \text{curl}\,\vec{E}$$

becomes with Maxwell's equations

$$-\text{div}\left[\vec{E} \times \vec{H}^*\right] = \vec{E} \cdot \left[\vec{J}^* + (-\alpha_1 - j\omega_1)D^*\right] + \vec{H}^* \cdot (-\alpha_1 + j\omega_1)\vec{B}$$

$$\begin{aligned}-\text{div}\left[\vec{E} \times \vec{H}^*\right] &= \vec{E} \cdot \vec{J}^* - \vec{E} \cdot (\alpha_1 + j\omega_1)\left[\left(\varepsilon + j\alpha_1 \frac{d\varepsilon}{d\omega}\right)^* \vec{E}^* + \left(\xi + j\alpha_1 \frac{d\xi}{d\omega}\right)^* \vec{H}^*\right] \\ &+ \vec{H}^* \cdot (-\alpha_1 + j\omega_1)\left[\left(\eta + j\alpha_1 \frac{d\eta}{d\omega}\right)\vec{E} + \left(\mu + j\alpha_1 \frac{d\mu}{d\omega}\right)\vec{H}\right]\end{aligned} \tag{2.29}$$

where all quantities ε, μ, ξ, η are taken at the real frequency ω_1.

To first order in α_1 we have

$$\begin{aligned}(\alpha_1 + j\omega_1)\left(\varepsilon + j\alpha_1 \frac{d\varepsilon}{d\omega}\right)^* &= (\alpha_1 + j\omega_1)\left(\varepsilon^* - j\alpha_1 \frac{d\varepsilon^*}{d\omega}\right) = \alpha_1\left(\varepsilon^* + \omega_1 \frac{d\varepsilon^*}{d\omega}\right) + j\omega_1\varepsilon^* - j\alpha_1^2 \frac{d\varepsilon^*}{d\omega} \\ &= \alpha_1 \frac{d}{d\omega}\left(\omega\varepsilon^*\right) + j\omega_1\varepsilon^* + \ldots\end{aligned} \tag{2.30}$$

$$(-\alpha_1 + j\omega_1)\left(\mu + j\alpha_1 \frac{d\mu}{d\omega}\right) = -\alpha_1\left(\mu + \omega_1 \frac{d\mu}{d\omega}\right) + j\omega_1\mu - j\alpha_1^2 \frac{d\mu}{d\omega} = -\alpha_1 \frac{d}{d\omega}(\omega\mu) + j\omega_1\mu \; +\ldots \quad (2.31)$$

Therefore

$$-\mathrm{div}\left[\vec{E}\times\vec{H}^*\right] = \vec{E}\cdot\vec{J}^* - \vec{E}\cdot\left[\alpha_1 \frac{d}{d\omega}\left(\omega\varepsilon^*\right) + j\omega_1\varepsilon^*\right]\vec{E}^* - \vec{E}\cdot\left[\alpha_1 \frac{d}{d\omega}\left(\omega\xi^*\right) + j\omega_1\xi^*\right]\vec{H}^*$$
$$+\vec{H}^*\cdot\left[-\alpha_1 \frac{d}{d\omega}(\omega\eta) + j\omega_1\eta\right]\vec{E} + \vec{H}^*\cdot\left[-\alpha_1 \frac{d}{d\omega}(\omega\mu) + j\omega_1\mu\right]\vec{H}$$

$$-\mathrm{div}\left[\vec{E}\times\vec{H}^*\right] = \vec{E}\cdot\vec{J}^* - \alpha_1\left\{\vec{E}\cdot\frac{d}{d\omega}\left(\omega\varepsilon^*\right)\vec{E}^* + \vec{E}\cdot\frac{d}{d\omega}\left(\omega\xi^*\right)\vec{H}^* + \vec{H}^*\cdot\frac{d}{d\omega}(\omega\eta)\vec{E} + \vec{H}^*\cdot\frac{d}{d\omega}(\omega\mu)\vec{H}\right\}$$
$$+ j\omega_1\left[-\vec{E}\cdot\varepsilon^*\vec{E}^* - \vec{E}\cdot\xi^*\vec{H}^* + \vec{H}^*\cdot\eta\vec{E} + \vec{H}^*\cdot\mu\vec{H}\right] \quad \text{or} \quad j\omega_1\left[-\vec{E}\cdot\vec{D}^*(\omega_1) + \vec{H}^*\cdot\vec{B}(\omega_1)\right] \quad (2.32)$$

Taking 1/2 Re of this expression yields the average power flowing into a unit volume:

$\frac{1}{2}\mathrm{Re}\left(\vec{E}\cdot\vec{J}^*\right)$ is the power delivered to the current $\vec{J}$;

$\frac{1}{2}\mathrm{Re}[-\alpha_1\{\ \}] = -2\alpha_1\frac{1}{4}\mathrm{Re}\{\ \} = \frac{\partial W}{\partial t}$ where $W = \frac{1}{4}\mathrm{Re}\{\ \}$ is the electromagnetic energy stored in a unit volume;

$\frac{1}{2}\mathrm{Re}\, j\omega_1\left[-\vec{E}\cdot\vec{D}^*(\omega_1) + \vec{H}^*\cdot\vec{B}(\omega_1)\right] = \frac{\omega_1}{2}\ \mathrm{Im}\left[-\vec{E}^*\cdot\vec{D}(\omega_1) - \vec{H}^*\cdot\vec{B}(\omega_1)\right]$ is the power dissipated as heat by hysteresis.

For the density of stored electromagnetic energy we thus obtain

$$W = \frac{1}{4}\mathrm{Re}\left\{\vec{E}^*\cdot\frac{d}{d\omega}\left(\omega\varepsilon^+\right)\vec{E} + \vec{H}^*\cdot\frac{d}{d\omega}(\omega\mu)\vec{H} + \vec{H}^*\cdot\frac{d}{d\omega}\left[\omega\left(\xi^+ + \eta\right)\right]\vec{E}\right\} \quad (2.33)$$

and for the power lost as heat by hysteresis:

$$P_{\text{hysteresis}} = \frac{\omega_1}{2}\mathrm{Im}\left\{-\vec{E}^*\cdot\left(\varepsilon\vec{E} + \xi\vec{H}\right) - \vec{H}^*\cdot\left(\eta\vec{E} + \mu\vec{H}\right)\right\} = \frac{\omega_1}{2}\mathrm{Im}\left\{-\vec{E}^*\cdot\varepsilon\vec{E} - \vec{H}^*\cdot\mu\vec{H} + \vec{H}^*\cdot\left(\xi^+ - \eta\right)\vec{E}\right\} \quad (2.34)$$

Let us remember that in (2.33) ω must be taken as real. Also, by taking the hermitian conjugate of the first term in (2.33) we see that

$$\mathrm{Re}\left\{\vec{E}^*\cdot\frac{d}{d\omega}\left(\omega\varepsilon^+\right)\vec{E}\right\} = \mathrm{Re}\left\{\vec{E}^*\cdot\frac{d}{d\omega}(\omega\varepsilon)\vec{E}\right\}$$

With $\xi = \eta = 0$, the expression (2.33) has been given by Landau and Lifshitz ([14], p. 255) and by Collin ([8], p. 28; [15], p. 16). When ε, μ are scalar quantities it becomes

$$W = W_e + W_m = \frac{1}{4}\frac{d}{d\omega}(\omega\varepsilon')\ \left|\vec{E}\right|^2 + \frac{1}{4}\frac{d}{d\omega}(\omega\mu')\ \left|\vec{H}\right|^2 \quad (2.35)$$

where W_e, W_m are the electric and magnetic energy densities.

This expression loses its validity in regions of anomalous dispersion ([14], p. 256) where it would yield negative values for W. At frequencies where dispersion is negligible, it reduces to the well known formula

$$W = W_e + W_m = \frac{1}{4}\varepsilon'\left|\vec{E}\right|^2 + \frac{1}{4}\mu'\left|\vec{H}\right|^2 \quad (2.36)$$

which is the time average of (2.26).

When ε, μ, ξ, η are scalar quantities the power loss by hysteresis (2.34) becomes

$$P_{\text{hysteresis}} = \frac{\omega_1}{2}\left\{\varepsilon''\left|\vec{E}\right|^2 + \mu''\left|\vec{E}\right|^2 + \operatorname{Im}\left[\left(\xi^+ - \eta\right)\vec{E}\cdot\vec{H}^*\right]\right\} \tag{2.37}$$

This expression must be positive for any field $\vec{E}$, $\vec{H}$; this requires that

$$\omega\varepsilon'' \geq 0 \qquad \omega\mu'' \geq 0 \qquad \varepsilon''\mu'' \geq \left|\frac{\xi^+ - \eta}{2}\right|^2 \tag{2.38}$$

In the following we will restrict ourselves to isotropic and homogeneous media where $\xi = \eta = 0$.

2.2 Boundary conditions at an interface between two different media

Normal components

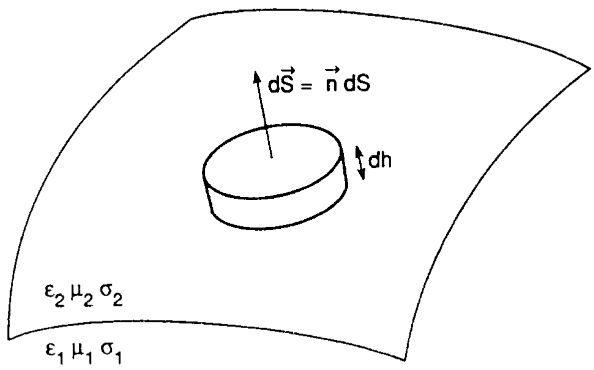


Fig. 2.2 Closed surface for div equations. $\vec{n}$ is a unit vector normal to the interface

Electric surface charge

$$\operatorname{div}\vec{D} = \rho \quad \text{(free electric charge)}$$

$$\vec{D}_2\cdot d\vec{S} - \vec{D}_1\cdot d\vec{S} = \rho_s dS$$

$$\boxed{\vec{n}\cdot\left(\vec{D}_2 - \vec{D}_1\right) = \rho_s \text{(free electric charge surface density)}} \tag{2.39}$$

Magnetic surface charge

$$\operatorname{div}\vec{B} = \rho_m$$

$$\boxed{\vec{n}\cdot\left(\vec{B}_2 - \vec{B}_1\right) = \rho_{ms} \text{ (free magnetic charge surface density)} \Rightarrow 0} \tag{2.40}$$

Tangential components

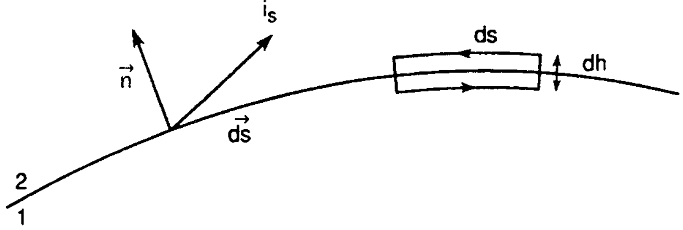


Fig. 2.3 Closed contour for curl equations. $\vec{i}_s$ is the surface density of electric current.

$$\operatorname{curl} \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{2.41}$$

With $\vec{1}_s$ representing a unit vector along s, this yields

$$\vec{H}_2 \cdot d\vec{s} - \vec{H}_1 \cdot d\vec{s} = \vec{i}_s ds \cdot \left[\vec{n} \times \vec{1}_s\right] = \vec{i}_s \cdot \left[\vec{n} \times d\vec{s}\right] = d\vec{s} \cdot \left[\vec{i}_s \times \vec{n}\right]$$

$$\boxed{\left(\vec{H}_2 - \vec{H}_1\right)_t = \left[\vec{i}_s \times \vec{n}\right] \quad \text{or} \quad \left[\vec{n} \times \left(\vec{H}_2 - \vec{H}_1\right)\right] = \vec{i}_s} \tag{2.42}$$

$$\operatorname{curl} \vec{E} = -\vec{J}_m - \frac{\partial \vec{B}}{\partial t} \tag{2.43}$$

$$\boxed{\left(\vec{E}_2 - \vec{E}_1\right)_t = -\left[\vec{i}_{ms} \times \vec{n}\right] \quad \text{or} \quad \left[\vec{n} \times \left(\vec{E}_2 - \vec{E}_1\right)\right] = -\vec{i}_{ms} \Rightarrow 0} \tag{2.44}$$

Remark. The condition

$$\left(\vec{H}_2 - \vec{H}_1\right)_t = \left[\vec{i}_s \times \vec{n}\right]$$

entails

$$\vec{n} \cdot \left(\operatorname{curl} \vec{H}_2 - \operatorname{curl} \vec{H}_1\right) = \vec{n} \cdot \operatorname{curl} \left[\vec{i}_s \times \vec{n}\right]$$

From the definition of curl, this is

$$\underbrace{\frac{1}{S} \oint \left[\vec{i}_s \times \vec{n}\right] \cdot d\vec{s}}_{\text{line integral of } \left[\vec{i}_s \times \vec{n}\right] \text{ along a closed loop of area } S}$$

which is also

$$\frac{1}{S} \oint \vec{i}_s \cdot \left[\vec{n} \times d\vec{s}\right] = -\operatorname{div} \vec{i}_s \ ,$$

the surface divergence of $\vec{i}_s$ (by definition of div). Combining with (2.41) we have

$$-\operatorname{div} \vec{i}_s = \vec{n} \cdot \left[\left(\vec{J}_2 - \vec{J}_1\right) + \frac{\partial}{\partial t}\left(\vec{D}_2 - \vec{D}_1\right)\right]$$

With (2.39) this reads

$$\boxed{\begin{array}{c} \operatorname{div} \vec{i}_s + \vec{n} \cdot \left(\vec{J}_2 - \vec{J}_1\right) + \dfrac{\partial \rho_s}{\partial t} = 0 \\ \text{Conservation of free electric charge at the interface} \end{array}} \tag{2.45}$$

From (2.43) there is a similar equation for the conservation of free magnetic charge at the interface.

In particular, if there is no free surface electric current along the interface,

$$\left(\vec{H}_2 - \vec{H}_1\right)_t = 0$$

entails

$$\vec{n}\cdot\left(\vec{J}_2-\vec{J}_1\right)+\frac{\partial}{\partial t}\vec{n}\cdot\left(\vec{D}_2-\vec{D}_1\right)=0$$

from Maxwell's equation (2.41). This relation means that the total current (conduction + displacement) is continuous when flowing through the interface. When rewritten as

$$\left(\sigma_2+j\omega\varepsilon_2\right)\ \vec{n}\cdot\vec{E}_2-\left(\sigma_1+j\omega\varepsilon_1\right)\ \vec{n}\cdot\vec{E}_1=0\ , \tag{2.46}$$

it also shows that even if there is no surface current, there is a free surface electric charge $\rho_s=\left(\varepsilon_2\vec{n}\cdot\vec{E}_2-\varepsilon_1\vec{n}\cdot\vec{E}_1\right)$ at the interface between two media with non zero conductivity, unless

$$\frac{\sigma_2}{\sigma_1}=\frac{\varepsilon_2}{\varepsilon_1}\ .$$

Similarly, since there is no free magnetic charge nor current along the interface,

$$\left(\vec{E}_2-\vec{E}_1\right)_t=0$$

always entails

$$\frac{\partial}{\partial t}\vec{n}\cdot\left(\vec{B}_2-\vec{B}_1\right)=0$$

from Maxwell's equation (2.43).

Therefore, when $\omega\neq 0$ it is sufficient to impose the continuity of the tangential components of $\vec{E}$ and $\vec{H}$; eventually the discontinuity of the normal component of $\vec{D}$ will yield the free surface electric charge.

Case when the medium (1) is infinitely conducting ($\sigma=\infty$)

From $\vec{J}=\sigma\vec{E}$ we have $\vec{E}=0$ inside the conductor. It is just sufficient to impose $\vec{E}_{2t}=0$ at the outer surface of the conductor. From (2.43) $\dfrac{\partial\vec{B}}{\partial t}=0$ inside the conductor. When $\omega\neq 0$ this requires $\vec{B}_1=\mu_1\vec{H}_1=0$; by (2.42) $\vec{H}_{2t}$ then determines the electric current $\vec{i}_s$ at the surface of the conductor.

2.3 Plane waves in isotropic, homogeneous and linear media

In such media ε, μ are scalar quantities independent of the position in space and of the field intensity.

$$\vec{J}=\sigma\vec{E}\qquad\qquad \vec{J}_m=0$$

From (2.2) $\qquad \operatorname{curl}\vec{H}=(\sigma+j\omega\varepsilon)\vec{E}\qquad \operatorname{curl}\vec{E}=-j\omega\mu\vec{H}\ .$

The other equations $\qquad \operatorname{div}\vec{E}=0\qquad \operatorname{div}\vec{H}=0$

are automatically satisfied.

The *Laplacian* vector operator ([16], p. 23)

$$\Delta\equiv\text{grad div}\ -\ \text{curl curl}$$

is such that $\qquad \left(\Delta\vec{E}\right)_\xi=\Delta\left(E_\xi\right)$

in cartesian coordinates, where $\xi=x,y,z$.

Maxwell's equations are equivalent to

$$\Delta \vec{E} = j\omega\mu(\sigma + j\omega\varepsilon)\vec{E} \quad \text{with} \quad \operatorname{div} \vec{E} = 0$$

with the same equation for $\vec{H}$.

Let

$$\boxed{k^2 = -j\omega\mu(\sigma + j\omega\varepsilon)} \tag{2.47}$$

with

$$\frac{\operatorname{Re}(k)}{\omega} > 0 \ ^{(*)}$$

The previous equation reads

$$\Delta \vec{E} + k^2 \vec{E} = 0 \quad \text{with} \quad \operatorname{div} \vec{E} = 0 \ . \tag{2.48}$$

The simplest solution to this equation is a plane wave

$$\vec{E} = \vec{E}_0 \ e^{-j\vec{k}\cdot\vec{r}} \tag{2.49}$$

where

$\vec{E}_0$ is a complex vector independent of the coordinates

$\vec{k}$ is a complex vector with components k_x, k_y, k_z

$$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$$

In cartesian coordinates, (2.48) reads $\Delta\left(E_\xi\right) + k^2 E_\xi = 0$

hence

$$(-jk_x)^2 + \left(-jk_y\right)^2 + (-jk_z)^2 + k^2 = 0$$

or

$$\boxed{k_x^2 + k_y^2 + k_z^2 = k^2} \tag{2.50}$$

The condition

$$\operatorname{div} \vec{E} = 0$$

reads

$$-jk_x E_x - jk_y E_y - jk_z E_z = 0$$

or

$$\boxed{\vec{k}\cdot\vec{E} = 0} \tag{2.51}$$

With (2.49), the magnetic field $\vec{H}$ can be obtained as

(*) The general expression for k^2 is

$$k^2 = -j\omega(\mu' - j\mu'')(\sigma + \omega\varepsilon'' + j\omega\varepsilon') = \left[\omega^2\varepsilon'\mu' - \omega\mu''(\sigma + \omega\varepsilon'')\right] - j\omega\left[\mu'(\sigma + \omega\varepsilon'') + \varepsilon'(\omega\mu'')\right]$$

In the transparency ranges where $\varepsilon' > 0$ and $\mu' > 0$, from (2.38) we have

$$\frac{\operatorname{Im}\left(k^2\right)}{\omega} = 2 \ \frac{\operatorname{Re}(k) \ \operatorname{Im}(k)}{\omega} \leq 0$$

Therefore the condition $\operatorname{Re}(k)/\omega > 0$ is equivalent to $\operatorname{Im}(k) \leq 0$.

$$-j\omega\mu\vec{H} = \text{curl}\,\vec{E} = \left[\text{grad}\;e^{-j\vec{k}\cdot\vec{r}} \times \vec{E}_0\right] = \left[\text{grad}\left(-j\vec{k}\cdot\vec{r}\right) \times \vec{E}\right]$$

$$= \left[-j\vec{k} \times \vec{E}\right] \qquad (2.52)$$

Therefore $$\vec{H} = \vec{H}_0\; e^{-j\vec{k}\cdot\vec{r}} \qquad (2.53)$$

and from (2.52)

$$\boxed{\vec{k}\cdot\vec{H} = 0 \qquad \vec{E}\cdot\vec{H} = 0} \qquad (2.54)$$

It should be remembered that $\vec{E}\cdot\vec{H} = 0$ does not necessarily mean that the physical electric and magnetic fields are orthogonal; but they are so if one of them is polarized linearly.

Reciprocally $$(\sigma + j\omega\varepsilon)\vec{E} = \text{curl}\,\vec{H} = \left[-j\vec{k} \times \vec{H}\right] \qquad (2.55)$$

Equations (2.52) and (2.55) can be rewritten as

$$\boxed{\vec{H} = \frac{jk}{j\omega\mu}\left[\vec{n} \times \vec{E}\right] \qquad \vec{E} = \frac{jk}{\sigma + j\omega\varepsilon}\left[\vec{H} \times \vec{n}\right]} \qquad (2.56)$$

where $\vec{k} = k\vec{n}$. From (2.50)

$$\boxed{n_x^2 + n_y^2 + n_z^2 = 1} \qquad (2.57)$$

which means that $\vec{n}$ is a unit vector; but n_x, n_y, n_z may be complex.

Equations (2.51) and (2.54) can be rewritten as

$$\boxed{\vec{n}\cdot\vec{E} = 0 \qquad \vec{n}\cdot\vec{H} = 0\;; \qquad \text{moreover}\;\; \vec{E}\cdot\vec{H} = 0} \qquad (2.58)$$

If $\vec{n}$ is real, the loci of constant phase (and amplitude) are planes defined by

$$\vec{n}\cdot\vec{r} = \text{constant}\;,$$

orthogonal to $\vec{n}$. Therefore the wave is plane, and the vector $\vec{n}$ yields the direction of propagation of the plane wave; (2.58) shows that these waves are transverse.

Propagation constant γ

It is such that $\vec{E} \sim e^{-\gamma\vec{n}\cdot\vec{r}}$; from (2.47) and (2.49) we see that

$$\gamma = \alpha + j\beta = jk = j\sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)} \qquad \text{where } \alpha \geq 0\;,\quad \frac{\beta}{\omega} > 0 \qquad (2.59)$$

When $\sigma = \varepsilon'' = \mu'' = 0$, $\alpha = 0$, $$\boxed{k = \beta = \omega\sqrt{\varepsilon\mu}} \qquad (2.60)$$

The phase velocity of the wave is $$\upsilon_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\varepsilon\mu}}\;. \qquad (2.61)$$

In free space, $$\upsilon_p = c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 2.99792458 \times 10^8\;\text{ms}^{-1} \qquad (2.62)$$

In MKSA units,

$$\mu_0 = 4\pi \cdot 10^{-7} \text{ Hm}^{-1} \text{ (by definition)}$$

$$\varepsilon_0 = 1/(\mu_0 c^2) \text{ is a derived quantity}$$

Intrinsic impedance ζ of a medium for plane waves

It is defined by the relation

$$\vec{E} = \zeta\left[\vec{H} \times \vec{n}\right] \quad \text{or} \quad \vec{H} = \frac{1}{\zeta}\left[\vec{n} \times \vec{E}\right] \tag{2.63}$$

From Eq. (2.56)

$$\boxed{\zeta = \frac{jk}{\sigma + j\omega\varepsilon} = \frac{j\omega\mu}{jk} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{\omega\mu'' + j\omega\mu'}{(\sigma + \omega\varepsilon'') + j\omega\varepsilon'}}} \quad \text{with } \mathrm{Re}(\zeta) > 0. \tag{2.64}$$

Using (2.59) we have

$$\zeta = \frac{j\omega\mu}{jk} = \frac{\omega\mu'' + j\omega\mu'}{\alpha + j\beta}$$

In the transparency ranges where $\varepsilon' > 0$ and $\mu' > 0$, (2.38) entails $\mathrm{Re}(\zeta) > 0$. The last expression in (2.64) also shows that

$$\zeta(-\omega) = \zeta^*(\omega) \quad \text{and} \quad \left|\arg\left(\zeta^2\right)\right| < \frac{\pi}{2}$$

hence

$$|\arg(\zeta)| < \frac{\pi}{4} \tag{2.65}$$

If $\vec{n} = (0, 0, 1)$, Eq. (2.63) reads $\zeta = E_x/H_y = -E_y/H_x$.

Impedance of free space

From (2.64) and (2.62), $\zeta_0 = \sqrt{\mu_0 / \varepsilon_0} = c \cdot \mu_0 \approx 30 \cdot 4\pi\ \Omega = 120\pi\ \Omega$.

Density of stored energies

From (2.36), if $\vec{n}$ is real,

$$\frac{W_e}{W_m} = \frac{\varepsilon'\left|\vec{E}^2\right|}{\mu'\left|\vec{H}^2\right|} = \frac{\varepsilon'}{\mu'}\left|\frac{j\omega\mu}{\sigma + j\omega\varepsilon}\right| = \left|\frac{\omega\varepsilon'\dfrac{\mu''}{\mu'} + j\omega\varepsilon'}{(\sigma + \omega\varepsilon'') + j\omega\varepsilon'}\right|$$

When the medium is lossless, $\sigma = \varepsilon'' = \mu'' = 0$ and $W_e = W_m$ in plane waves.

2.4 Reflection and refraction of a plane wave at the interface between two different media

If there is a surface current $\vec{i}_s$, we may consider it as a limit $\vec{i}_s = \vec{J}\,dz$ when the depth $dz \to 0$ and $\vec{J} \to \infty$. But then the Joule power loss per unit area of the surface would be

$$P_1 = \frac{1}{2\sigma}\left|\vec{J}\right|^2 dz = \frac{1}{2\sigma\ dz}\left|\vec{i}_s\right|^2 .$$

For P_1 remaining finite when $dz \rightarrow 0$, we must have $\vec{i}_s = 0$: from Eq. (2.42) $\vec{H}_t$ is then continuous at the interface. The only possible case for a non-zero surface current is when $\sigma = \infty$.

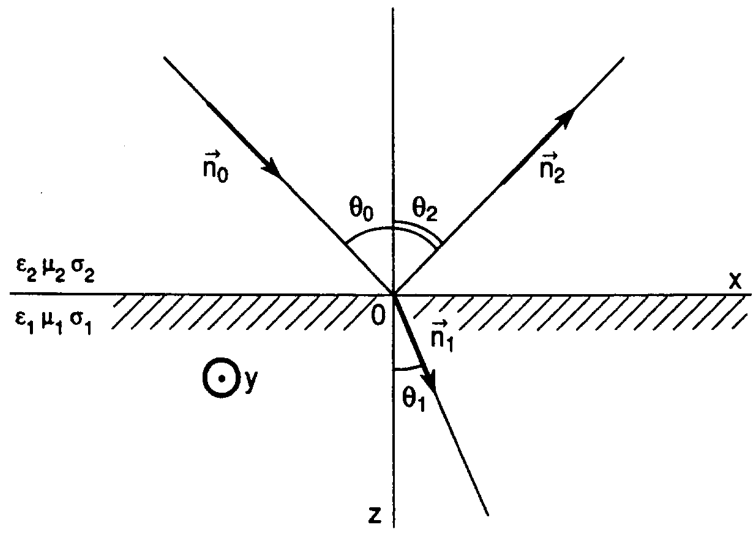


Fig. 2.4 Plane interface between two different media

Incident plane wave

Referring to Fig. 2.4, following (2.49), (2.51) and (2.63) the incident wave in medium 2 reads

$$\vec{E}_I = \vec{E}_0\ e^{-jk_2\vec{n}_0\cdot\vec{r}}\ ,\quad \vec{n}_0\cdot\vec{E}_I = 0\ ,\quad \vec{H}_I = \frac{1}{\zeta_2}\left[\vec{n}_0\times\vec{E}_I\right] \tag{2.66}$$

Reflected plane wave

The reflected wave in medium 2 reads

$$\vec{E}_R = \vec{E}_2\ e^{-jk_2\vec{n}_2\cdot\vec{r}}\ ,\quad \vec{n}_2\cdot\vec{E}_R = 0\ ,\quad \vec{H}_R = \frac{1}{\zeta_2}\left[\vec{n}_2\times\vec{E}_R\right] \tag{2.67}$$

Transmitted plane wave

The transmitted wave in medium 1 reads

$$\vec{E}_T = \vec{E}_1\ e^{-jk_1\vec{n}_1\cdot\vec{r}}\ ,\quad \vec{n}_1\cdot\vec{E}_T = 0\ ,\quad \vec{H}_T = \frac{1}{\zeta_1}\left[\vec{n}_1\times\vec{E}_T\right] \tag{2.68}$$

Boundary conditions: Continuity of tangential components of $\vec{E}$ and $\vec{H}$ at $z = 0$

$$\vec{E}_{It} + \vec{E}_{Rt} = \vec{E}_{Tt} \qquad \vec{H}_{It} + \vec{H}_{Rt} = \vec{H}_{Tt} \tag{2.69}$$

Determination of $\vec{n}_1$ and $\vec{n}_2$

In order to satisfy the boundary conditions at every point of the interface, the arguments of the exponentials must be identical at $z = 0$:

$$k_2\left(n_{0x}x + n_{0y}y\right) = k_2\left(n_{2x}x + n_{2y}y\right) = k_1\left(n_{1x}x + n_{1y}y\right) ,$$

hence

$$\boxed{\begin{array}{l|l} n_{0x} = n_{2x} & k_2 n_{0x} = k_1 n_{1x} \\ n_{0y} = n_{2y} & k_2 n_{0y} = k_1 n_{1y} \end{array}} \qquad (2.70)$$

where, by (2.57):

$$n_{0x}^2 + n_{0y}^2 + n_{0z}^2 = n_{2x}^2 + n_{2y}^2 + n_{2z}^2 = n_{1x}^2 + n_{1y}^2 + n_{1z}^2 = 1 .$$

Therefore

$$n_{0z}^2 = n_{2z}^2 \quad \text{and} \quad \boxed{k_2^2\left(1 - n_{0z}^2\right) = k_1^2\left(1 - n_{1z}^2\right) = k_1^2 - \left(k_1 n_{1z}\right)^2} \qquad (2.71)$$

In order to distinguish the reflected wave from the incident wave, we must take

$$n_{2z} = -n_{0z}. \qquad (2.72)$$

However, the branch of n_{1z} cannot be determined from the boundary conditions, because both determinations of n_{1z} satisfy these conditions. In order to specify that the transmitted wave is receding from the interface, we must take

$$\mathrm{Im}\left(k_1 n_{1z}\right) < 0 \quad \text{or} \quad \mathrm{Re}\left(k_1 n_{1z}\right) > 0 \quad \text{when} \quad \mathrm{Im}\left(k_1 n_{1z}\right) = 0 . \qquad (2.73)$$

On physical grounds, such conditions are also satisfied by the incident and reflected waves:

$$\mathrm{Im}\left(k_2 n_{0z}\right) < 0 \qquad \mathrm{Im}\left(k_2 n_{2z}\right) > 0 \qquad (2.74)$$

or

$$\mathrm{Re}\left(k_2 n_{0z}\right) > 0 \quad \text{when} \quad \mathrm{Im}\left(k_2 n_{0z}\right) = 0 \; ; \quad \mathrm{Re}\left(k_2 n_{2z}\right) < 0 \quad \text{when} \quad \mathrm{Im}\left(k_2 n_{2z}\right) = 0$$

The above relations (2.70 to 2.73) giving $\vec{n}_1$, $\vec{n}_2$, contain all the laws of reflection and refraction.

Determination of the tangential components of the reflected and transmitted waves at the interface of two media.

Relation between the tangential components of $\vec{E}$ and $\vec{H}$

Using (2.58) and (2.63) it is possible to write for any plane wave

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = Z \begin{bmatrix} H_x \\ H_y \end{bmatrix} \quad \text{or at } z = 0 : \quad E_t = Z \; H_t \qquad (2.75)$$

where Z is a 2 x 2 matrix which we now determine.

From (2.63) and (2.58):

$$E_x = \zeta\left[H_y n_z - H_z n_y\right] = \frac{\zeta}{n_z}\left[H_y n_z^2 + \left(H_x n_x + H_y n_y\right) n_y\right] = \frac{\zeta}{n_z}\left[H_x n_x n_y + H_y\left(n_y^2 + n_z^2\right)\right]$$

$$E_y = \zeta\left[H_z n_x - H_x n_z\right] = \frac{\zeta}{n_z}\left[-\left(H_x n_x + H_y n_y\right) n_x - H_x n_z^2\right] = \frac{\zeta}{n_z}\left[-H_x\left(n_x^2 + n_z^2\right) - H_y n_x n_y\right]$$

Remembering (2.57) we thus have

$$Z = \frac{\zeta}{n_z}\begin{bmatrix} n_x n_y & \left(n_y^2 + n_z^2\right) \\ -\left(n_x^2 + n_z^2\right) & -n_x n_y \end{bmatrix} = \frac{\zeta}{n_z}\begin{bmatrix} n_x n_y & \left(1 - n_x^2\right) \\ -\left(1 - n_y^2\right) & -n_x n_y \end{bmatrix} \tag{2.76}$$

It is easy to verify that

$$Z^2 = -\zeta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{hence} \quad Z^{-1} = -\frac{1}{\zeta^2} Z$$

Reflection and transmission matrices

For the tangential electric field, these 2 x 2 matrices R, T are defined by the relations

$$E_{2t} = R \ E_{0t} \qquad E_{1t} = T \ E_{0t} \tag{2.77}$$

The boundary conditions (2.69) entail that

$$T = I + R \tag{2.78}$$

and that

$$Z_0 H_{0t} + Z_2 H_{2t} = Z_1 H_{1t} \quad \text{with} \quad H_{0t} + H_{2t} = H_{1t}$$

Using (2.70) and (2.72) in (2.76) we notice that $Z_2 = -Z_0$.

Therefore

$$2Z_0 H_{0t} = (Z_1 + Z_0) H_{1t}.$$

For the magnetic field:

$$H_{1t} = \left(Z_1 + Z_0\right)^{-1} 2 Z_0 H_{0t} \tag{2.79}$$

For the electric field:

$$E_{1t} = Z_1 H_{1t} = 2 Z_1 \left(Z_1 + Z_0\right)^{-1} E_{0t} \tag{2.80}$$

With (2.77) and (2.37) this yields

$$\begin{matrix} T = 2Z_1 \left(Z_1 + Z_0\right)^{-1} & R = \left(Z_1 - Z_0\right)\left(Z_1 + Z_0\right)^{-1} \\ T^{-1} = \left(Z_1 + Z_0\right)\left(2Z_1\right)^{-1} & R^{-1} = \left(Z_1 + Z_0\right)\left(Z_1 - Z_0\right)^{-1} \end{matrix} \tag{2.81}$$

Eigenvectors of R and T

Let us write (2.76) as

$$\frac{n_z}{\zeta} Z = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} n_x n_y & -n_x^2 \\ n_y^2 & -n_x n_y \end{bmatrix}$$

Let

$$U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{hence} \quad U^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.82}$$

Then

$$\frac{n_z}{\zeta} ZU = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} n_x^2 & n_x n_y \\ n_x n_y & n_y^2 \end{bmatrix}$$

It is straightforward to see that the eigenvectors of the last matrix, which are also eigenvectors of ZU, are

$$\begin{bmatrix} n_y \\ -n_x \end{bmatrix} \text{ and } \begin{bmatrix} n_x \\ n_y \end{bmatrix} \tag{2.83}$$

Remembering (2.57) we have

$$\frac{n_z}{\zeta} ZU\begin{bmatrix} n_y \\ -n_x \end{bmatrix} = -\begin{bmatrix} n_y \\ -n_x \end{bmatrix} \qquad \frac{n_z}{\zeta} ZU\begin{bmatrix} n_x \\ n_y \end{bmatrix} = \left(-1 + n_x^2 + n_y^2\right)\begin{bmatrix} n_x \\ n_y \end{bmatrix} = -n_z^2\begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

i.e.

$$ZU\begin{bmatrix} n_y \\ -n_x \end{bmatrix} = -\frac{\zeta}{n_z}\begin{bmatrix} n_y \\ -n_x \end{bmatrix} \qquad ZU\begin{bmatrix} n_x \\ n_y \end{bmatrix} = -\zeta n_z \begin{bmatrix} n_x \\ n_y \end{bmatrix} \tag{2.84}$$

The expression (2.81) for R can be rewritten as

$$R = (Z_1 - Z_0)UU^{-1}(Z_1 + Z_0)^{-1} = (Z_1U - Z_0U)(Z_1U + Z_0U)^{-1} \tag{2.85}$$

From (2.70) it is obvious that both eigenvectors are the same for the three waves 0, 1, 2. This means that the eigenvectors for the incident wave are simultaneously eigenvectors of Z_0U, Z_1U and therefore they are also eigenvectors of $(Z_1U{-}Z_0U)$, $(Z_1U{+}Z_0U)$, $(Z_1U{+}Z_0U)^{-1}$ and finally of R, T. From (2.84) and (2.85) we obtain the eigenvalues ρ_1, ρ_2 of the reflection matrix R as:

1) for the eigenvector

$$\begin{bmatrix} E_{ox} \\ E_{oy} \end{bmatrix} \sim \begin{bmatrix} n_{oy} \\ -n_{ox} \end{bmatrix}$$

i.e. when $n_{ox}E_{ox} + n_{oy}E_{oy} = 0$ (with similar relations for waves 1, 2):

$$\rho_1 = \left(\frac{\zeta_1}{n_{1z}} - \frac{\zeta_2}{n_{0z}}\right)\left(\frac{\zeta_1}{n_{1z}} + \frac{\zeta_2}{n_{0z}}\right)^{-1} \tag{2.86}$$

2) for the eigenvector

$$\begin{bmatrix} E_{ox} \\ E_{oy} \end{bmatrix} \sim \begin{bmatrix} n_{ox} \\ n_{oy} \end{bmatrix}$$

i.e. when $n_{ox}E_{oy} - n_{oy}E_{ox} = 0$ (with similar relations for waves 1, 2):

$$\rho_2 = (\zeta_1 n_{1z} - \zeta_2 n_{0z})(\zeta_1 n_{1z} + \zeta_2 n_{0z})^{-1} \tag{2.87}$$

Physical interpretation of the eigenvectors of R, T

They correspond to particular polarizations of the incident plane wave.

Case 1

For the three waves 0, 1, 2 we have

$$n_x E_x + n_y E_y = 0 \tag{2.88}$$

With (2.58) this entails $\boldsymbol{E_z = 0}$ for all waves.

If the ratio n_{oy}/n_{ox} is real, the relation (2.88) means that the electric field is normal to the plane of incidence.

Case 2

For the three waves 0, 1, 2 we have

$$n_x E_y - n_y E_x = 0 \tag{2.89}$$

With (2.63) this entails $\boldsymbol{H_z = 0}$ for all waves.

If the ratio n_{oy}/n_{ox} is real, the relation (2.89) means that the electric field lies in the plane of incidence.

The eigenvalues of the reflection matrix R, i.e. the reflection coefficients ρ are different for the two cases. They are given by Fresnel formulae (2.86) and (2.87), which Fresnel derived for real ζ and $\vec{n}$.

Remarks

1) Considering the propagation along z, the two polarizations correspond respectively to TE ($E_z = 0$) and TM ($H_z = 0$) waves.

2) For the magnetic field, from (2.79) and (2.78) we obtain the transmission and reflection matrices as

$$T_H = (Z_1 + Z_0)^{-1} 2Z_0 \qquad R_H = -(Z_1 + Z_0)^{-1}(Z_1 - Z_0) \tag{2.90}$$

Therefore

$$U^{-1} R_H U = -U^{-1}(Z_1 + Z_0)^{-1}(Z_1 - Z_0)U = -(Z_1 U + Z_0 U)^{-1}(Z_1 U - Z_0 U) \tag{2.91}$$

The eigenvectors (2.83) of ZU are thus also eigenvectors of $U^{-1}R_H U$, which means that

$$U\begin{bmatrix} n_y \\ -n_x \end{bmatrix} = -\begin{bmatrix} n_x \\ n_y \end{bmatrix} \quad \text{and} \quad U\begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} n_y \\ -n_x \end{bmatrix} \tag{2.92}$$

are eigenvectors of R_H, with the same eigenvalues as (2.91). Comparing with (2.85) we see that the eigenvalues of R_H are the same as the eigenvalues of R, but opposite in sign: the reflection coefficients for the tangential magnetic field are opposite in sign to the reflection coefficients for the tangential electric field.

Surface impedance

At the interface $z = 0$ the relation between the tangential components of the fields is given by the matrix equation (2.75) for each one of the incident, reflected and transmitted wave.

1) TE *wave*

When H_t is the first eigenvector (2.92), by (2.84) this relation simplifies to

$$E_t = ZU\begin{bmatrix} n_y \\ -n_x \end{bmatrix} = -\frac{\zeta}{n_z}\begin{bmatrix} n_y \\ -n_x \end{bmatrix} \quad \text{with} \quad H_t = U\begin{bmatrix} n_y \\ -n_x \end{bmatrix}$$

Using (2.82) this can be rewritten as

$$E_t = \frac{\zeta}{n_z} U^2 \begin{bmatrix} n_y \\ -n_x \end{bmatrix} = \frac{\zeta}{n_z} U \; H_t$$

or in vector form:

$$\vec{E}_t = Z_s \left[\vec{H}_t \times \vec{1}_z \right] \quad \text{where} \quad Z_s = \frac{\zeta}{n_z} \tag{2.93}$$

Considering the propagation along z, this relation is analogous to (2.63); Z_s is called the *surface impedance* for either the incident, reflected or transmitted wave.

For the incident wave, $$Z_{0s} = \frac{\zeta_2}{n_{0z}}$$

For the reflected wave, $$Z_{2s} = \frac{\zeta_2}{n_{2z}} = -Z_{0s}$$

For the transmitted wave, $$Z_{1s} = \frac{\zeta_1}{n_{1z}}$$

The reflection coefficient (2.86) then reads

$$\rho = (Z_{1s} - Z_{0s})(Z_{1s} + Z_{0s})^{-1} \tag{2.94}$$

which is identical to the reflection coefficient in a waveguide, as given later in (4.21).

2) TM *wave*

When H_t is the second eigenvector (2.92), all the above still applies with ζ/n_z simply replaced by ζn_z; in particular we have

$$Z_s = \zeta n_z \tag{2.95}$$

Laws of reflection and refraction

When the ratio n_{oy}/n_{ox} is real, it is possible by rotating the coordinate system about the z–axis, to choose it so that $n_{oy} = 0$; the plane $y = 0$ is then the plane of incidence. When the media are lossless and $\vec{n}_o$ is real, from Fig. 2.4 we have

$$\begin{array}{lll} n_{0x} = \sin\theta_0 & n_{2x} = \sin\theta_2 & n_{1x} = \sin\theta_1 \\ n_{0y} = 0 & n_{2y} = 0 & \mathrm{n}_{1y} = 0 \\ n_{0z} = \cos\theta_0 & n_{2z} = -\cos\theta_2 & n_{1z} = \cos\theta_1 \end{array} \tag{2.96}$$

Equations (2.70) and (2.72) then yield

$$\theta_2 = \theta_0 \qquad \sqrt{\varepsilon_2\mu_2}\sin\theta_0 = \sqrt{\varepsilon_1\mu_1}\sin\theta_1 \tag{2.97}$$

which are the laws of reflection and refraction.

Total reflection at a dielectric interface

We assume that

$$n_{0y} = n_{1y} = n_{2y} = 0$$

There is total reflection if the transmitted wave in medium 1 decays exponentially from the interface; that is if $n_{1z}^2 < 0$. From (2.70) we have

$$k_x^2 = k_2^2 n_{0x}^2 = k_1^2 n_{1x}^2$$

or with (2.57):

$$k_x^2 = k_2^2\left(1 - n_{0z}^2\right) = k_1^2\left(1 - n_{1z}^2\right)$$

where

$$n_{0z}^2 > 0 , \qquad n_{1z}^2 < 0 .$$

Therefore

$$\boxed{k_1^2 < k_x^2 < k_2^2} \tag{2.98}$$

With (2.96) this is only possible if

$$\frac{k_2^2}{k_1^2} = \frac{\varepsilon_2\mu_2}{\varepsilon_1\mu_1} > \frac{1}{\sin^2\theta_0} > 1 \tag{2.99}$$

This property is used in dielectric waveguides. Light seems to penetrate at some distance from the interface in medium 1: it is the Goos-Hänchen shift ([17], p. 25). From (2.71) and (2.96) the penetration depth (analogous to the skin depth in good conductors) is the inverse of

$$\mathrm{Re}\left(jk_1 n_{1z}\right) = \mathrm{Re}\, j\sqrt{k_1^2 - k_2^2\left(1 - n_{0z}^2\right)} = \mathrm{Re}\,\sqrt{k_2^2\,\sin^2\theta_0 - k_1^2}$$

but the *effective* penetration depth is the inverse of

$$q \cdot \sqrt{k_2^2 \sin^2\theta_0 - k_1^2}$$

where q is a correction factor ([17], p. 27). For TE waves, $q = 1$; for TM waves, assuming $\mu_1 = \mu_2$,

$$q = \frac{k_2^2}{k_1^2}\sin^2\theta_0 - \cos^2\theta_0 > 0$$

When does the incident or the reflected wave vanish?

The formulae (2.80) presuppose that det $(Z_1 + Z_0) \neq 0$. If det $(Z_1 + Z_0) = 0$, det T^{-1} = det $R^{-1} = 0$. From (2.76) it is then possible to have $E_{2t} \neq 0$ and $E_{1t} \neq 0$ whilst $E_{0t} = 0$: in this case the incident wave vanishes.

If det $(Z_1 - Z_0) = 0$, det $R = 0$. It is then possible to have $E_{2t} = 0$ whilst $E_{0t} \neq 0$: in this case the reflected wave vanishes.

The equivalent conditions det $(Z_1U \pm Z_0U) = 0$ amount to saying that one of the eigenvalues of $(Z_1U \pm Z_0U)$ is zero; from (2.83) this reads

$$\frac{\zeta_1}{n_{1z}} \pm \frac{\zeta_2}{n_{0z}} = 0 \quad \text{for TE waves}, \qquad \zeta_1 n_{1z} \pm \zeta_2 n_{0z} = 0 \quad \text{for TM waves}.$$

With (2.71) this can be rewritten as

$$\frac{\zeta_1}{n_{1z}} - \frac{\zeta_2}{n_z} = 0 \quad \text{for TE waves}, \qquad \zeta_1 n_{1z} - \zeta_2 n_z = 0 \quad \text{for TM waves}. \tag{2.100}$$

where n_z represents either n_{2z} or n_{0z}, according to which one of the reflected or incident waves exists.

Case 1: TE waves

With (2.71) and (2.72) the first condition (2.100) transforms into

$$\frac{k_1^2\zeta_1^2}{(k_1 n_{1z})^2} = \frac{k_1^2\zeta_1^2}{k_1^2 - k_2^2(1-n_z^2)} = \frac{k_2^2\zeta_2^2}{(k_2 n_z)^2}$$

where by (2.64) $k\zeta = \omega\mu$.

Therefore

$$\frac{\mu_1^2}{\mu_2^2} = \frac{k_1^2 - k_2^2 + (k_2 n_z)^2}{(k_2 n_z)^2} \quad \text{or} \quad \frac{k_1^2 - k_2^2}{(k_2 n_z)^2} = \frac{\mu_1^2}{\mu_2^2} - 1 \tag{2.101}$$

Provided $\mu_1 \neq \mu_2$, the condition (2.101) determines $(k_2 n_z)^2$. The branch of $k_2 n_z$ is then chosen such as to satisfy the first equation (2.100); from (2.74) n_z is then interpreted as being n_{0z} if Im $(k_2 n_z) < 0$ or Re $(k_2 n_z) > 0$ when Im$(k_2 n_z) = 0$, and being n_{2z} if Im $(k_2 n_z) > 0$ or Re $(k_2 n_z) < 0$ when Im $(k_2 n_z) = 0$.

Case 2: TM waves

With (2.71) and (2.72) the second condition (2.100) transforms into

$$\frac{\zeta_1^2}{k_1^2}(k_1 n_{1z})^2 = \frac{\zeta_1^2}{k_1^2}\left[k_1^2 - k_2^2(1-n_z^2)\right] = \frac{\zeta_2^2}{k_2^2}(k_2 n_z)^2$$

where by (2.64) $\zeta/k = j/(\sigma + j\omega\varepsilon)$.

Therefore

$$\frac{k_1^2 - k_2^2}{(k_2 n_z)^2} = \frac{(\sigma_1 + j\omega\varepsilon_1)^2}{(\sigma_2 + j\omega\varepsilon_2)^2} - 1 \tag{2.102}$$

Provided $(\sigma_1 + j\omega\varepsilon_1) \neq (\sigma_2 + j\omega\varepsilon_2)$, the condition (2.102) determines $(k_2 n_z)^2$. The branch of $k_2 n_z$ is then chosen such as to satisfy the second equation (2.100); following the same procedure as above, n_z is then interpreted as being either n_{0z} or n_{2z}.

When $\mu_1 = \mu_2$, the wave is a TM wave; Eq. (2.47) then converts (2.102) into

$$\frac{1}{n_z^2} = \frac{\sigma_1 + j\omega\varepsilon_1}{\sigma_2 + j\omega\varepsilon_2} + 1$$

If n_z is n_{0z}, the wave is incident from $z = -\infty$; the particular value of n_{0z} determines an incidence angle (*Brewster angle*, real or complex) for which the reflected wave vanishes. When both media are lossless and $\mu_1 = \mu_2$, using (2.96) it is easy to see that (2.102) yields

$$\tan^2\theta_0 = \frac{\varepsilon_1}{\varepsilon_2} \tag{2.103}$$

It should be noticed that when both media are lossless, by (2.64) ζ_1, ζ_2 are real and positive; from (2.100) and (2.73), (2.74) it appears that n_z must be n_{0z}.

If n_z is n_{2z}, by the previous remark one at least of the two media is lossy; from (2.101) or (2.102) $k_2 n_{2z}$ is complex, as is also $k_1 n_{1z}$ by (2.71). There is no incident wave; both reflected and transmitted waves propagate along the interface and decay exponentially on both sides along the normal to the interface: they form what is called a *surface wave* ([6], p. 516; [15], p. 697).

Transmitted wave inside a good conductor

In Fig. 2.4, we suppose that $\sigma_2 = 0$; σ_1 is considered to be given by (2.18) in order to include the case of very high frequencies. From (2.47) we then have

$$k_2 = \omega\sqrt{\varepsilon_2\mu_2} \qquad k_1 = \sqrt{-j\omega\mu_1\left(\frac{\sigma_1}{1+j\omega\tau_1} + j\omega\varepsilon_1\right)} \quad \text{with} \quad \mathrm{Im}(k_1) \le 0 \tag{2.104}$$

In (2.104) and in the following, σ_1 represents the D.C. conductivity.

Orders of magnitude. For copper at room temperature, $\sigma_1 = 5.8 \times 10^7\ \Omega^{-1}\ \mathrm{m}^{-1}$, $\tau_1 = 2.4 \times 10^{-14}$ s; we take $\varepsilon_1 \approx \varepsilon_0$ because we are neglecting the effects of the bound electrons in the metal ([7], p. 32–10). Using (2.18) we can write

$$\frac{\sigma_1}{1+j\omega\tau_1} + j\omega\varepsilon_1 \approx j\omega\varepsilon_0 n^2 \quad \text{hence} \quad k_1 = \sqrt{\omega^2\varepsilon_0\mu_1 n^2} \tag{2.105}$$

where

$$n^2 = \frac{\sigma_1\tau_1/\varepsilon_0}{j\omega\tau_1(1+j\omega\tau_1)} + 1 \quad \text{and} \quad \frac{\sigma\tau}{\varepsilon_0} = (\omega_p\tau)^2 \tag{2.106}$$

Therefore

$$(\omega_p\tau)^2 = 1.6\cdot 10^5 \qquad \omega_p\tau = 400 \qquad f_p = \frac{\omega_p}{2\pi} = 2.6\cdot 10^{15}\,\mathrm{Hz} \quad (\lambda_p = 0.115\ \mu\mathrm{m})$$

We define a critical frequency f_c such that

$$\omega_c\tau = 1 \qquad f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi\tau} = 6.6\cdot 10^{12}\ \mathrm{Hz} \quad (\lambda_c = 45\mu\mathrm{m})$$

We have to consider three frequency ranges for which (2.106) and (2.105) take the following approximate forms:

1) when

$$f < f_c\ , \quad \omega\tau < 1\ ,$$
$$n^2 \approx \frac{\sigma_1}{j\omega\varepsilon_0} + 1 \approx \frac{\sigma_1}{j\omega\varepsilon_0}\ , \qquad k_1 \approx \sqrt{-j\omega\mu_1\sigma_1} \tag{2.107}$$

because at the upper frequency f_c, from (2.106) the modulus of the first term of n^2 is still $(\omega_p\tau)^2/\sqrt{2} = 1.13\cdot 10^5$. This means that in this frequency range (which includes the microwave range), the displacement current in the metal is completely negligible with respect to the conduction current.

2) when

$$f_c < f < f_p\ , \quad 1 < \omega\tau < \omega_p\tau\ ,$$
$$n^2 \approx 1 - \frac{\omega_p^2}{\omega^2}\ , \qquad k_1 \approx \sqrt{(\omega^2 - \omega_p^2)\varepsilon_0\mu_0} \tag{2.108}$$

The fields are attenuated exponentially in the metal, with a decay length of the order of $\left(\omega_p\sqrt{\varepsilon_0\mu_0}\right)^{-1} = c/\omega_p = 18$ nm. In copper at room temperature, the mean free path of conduction electrons is 42 nm ([18], p. 238) which is larger than the skin depth; in that case the local form (2.17) of Ohm's law is no longer valid, and the skin effect becomes anomalous ([12], p. 308; [18], p. 308). This frequency range is also the range where resonances of the bound electrons show up in ε_1.

3) when $f > f_p$ (i.e. $\lambda < 115$ nm), Eq. (2.108) still applies but now $\omega > \omega_p$ and k_1 is real: the metal becomes transparent. When the wavelength ultimately becomes comparable with the mesh size of the crystal lattice, the metal can no longer be considered as a continuous medium; this is the frequency range of X-rays, which is dominated by diffraction phenomena.

Coming back to the microwave range, we rewrite (2.107) as

$$k_1 = \sqrt{-j\omega\mu_1\sigma_1} = [\mathrm{sgn}(\omega) - j]\sqrt{\frac{|\omega|\mu_1\sigma_1}{2}} \quad \text{or} \quad jk_1 = [1 + j\,\mathrm{sgn}(\omega)]\sqrt{\frac{|\omega|\mu_1\sigma_1}{2}} \qquad (2.109)$$

From (2.47) we have

$$\frac{k_2^2}{k_1^2} = \frac{\omega^2\varepsilon_2\mu_2}{-j\omega\mu_1\sigma_1} = \frac{j\omega\varepsilon_0}{\sigma_1}\frac{\varepsilon_2\mu_2}{\varepsilon_0\mu_1} \quad \text{and} \quad \left|\frac{k_2}{k_1}\right|^2 \approx \frac{|\omega|\varepsilon_0}{\sigma_1} \approx f_{\mathrm{Hz}}\cdot 10^{-18} << 1\ , \qquad (2.110)$$

the last figure being for copper at 20° C.

Along z, the propagation constant of the transmitted wave is, with (2.71) and (2.73):

$$jk_1 n_{1z} = j\sqrt{k_1^2 - k_2^2\left(1 - n_{0z}^2\right)} \approx jk_1 \qquad (2.111)$$

Whatever is n_{0z}, the transmitted wave in the conductor varies as

$$\mathrm{e}^{-jk_1 z} = \mathrm{e}^{-[1+j\ \mathrm{sgn}(\omega)]\sqrt{\frac{|\omega|\mu_1\sigma_1}{2}}z} = \mathrm{e}^{-[1+j\ \mathrm{sgn}(\omega)]\frac{z}{\delta}}$$

where

$$\boxed{\delta = \sqrt{\frac{2}{|\omega|\mu_1\sigma_1}}} \quad \text{is the skin depth.} \qquad (2.112)$$

For copper at 20 °C, $\delta = 66.09\ \mu\mathrm{m}\ [f_{\mathrm{MHz}}]^{-1/2}$.

f δ $\left\|\frac{k_2}{k_1}\right\| = \sqrt{\frac{\|\omega\|\varepsilon}{\sigma}}$	50 Hz 9.35 mm $6.92\cdot 10^{-9}$	1 kHz 2.09 mm $3.09\cdot 10^{-8}$	1 MHz 66.1 μm $0.979\cdot 10^{-6}$	1 GHz 2.09 μm $3.09\cdot 10^{-5}$	100 GHz 0.21 μm $3.09\cdot 10^{-4}$

From (2.58)

$$\vec{n}_1 \cdot \vec{E}_1 = 0 \quad \text{or} \quad n_{1x}E_{1x} + n_{1y}E_{1y} + n_{1z}E_{1z} = 0$$

where by (2.70) and (2.73)

$$n_{1x} = \frac{k_2}{k_1}n_{0x}\ , \quad n_{1y} = \frac{k_2}{k_1}n_{0y}\ , \quad n_{1z} \approx 1\ , \quad \text{i.e. } \vec{n}_1 \approx \vec{1}_z \qquad (2.113)$$

Therefore $|E_{1z}| << |E_{1x}|$ or $|E_{1y}|$; the same applies for $\vec{H}_1$. This means that inside a good conductor, the fields are almost parallel to the conductor surface.

Surface impedance of a good conductor

Because $n_{1z} \approx 1$, the surface impedances (2.93) and (2.95) are practically the same for both TE and TM transmitted waves. Therefore, at the surface of the metal, the boundary condition (2.93):

$$\vec{E}_t = Z_s\left[\vec{H}_t \times \vec{1}_z\right] = Z_s\left[\vec{H}_t \times \vec{n}\right] \tag{2.114}$$

applies with the same value $Z_s = \zeta_1$ for any wave polarization; $\vec{n} = \vec{1}_z$ is a unit vector on the inward normal to the metal surface.

From (2.64) and (2.110), the surface impedance of the metal is thus

$$\boxed{Z_s = \zeta_1 \approx \sqrt{\frac{j\omega\mu_1}{\sigma_1}} = [1 + j\ \mathrm{sgn}(\omega)]R_s \qquad \text{where} \qquad R_s = \sqrt{\frac{|\omega|\mu_1}{2\sigma_1}}\ .} \tag{2.115}$$

Since

$$Z_s = \sqrt{\frac{\mu_1}{\varepsilon_1}}\sqrt{\frac{j\omega\varepsilon_1}{\sigma_1}}\ , \qquad |Z_s| << \sqrt{\frac{\mu_1}{\varepsilon_1}}\ .$$

Using (2.63), (2.113) and (2.115) we see that inside the metal

$$\frac{W_e}{W_m} = \frac{\varepsilon_1\left|\vec{E}_1\right|^2}{\mu_1\left|\vec{H}_1\right|^2} \approx \frac{\varepsilon_1}{\mu_1}|\zeta_1|^2 \approx \frac{|\omega|\varepsilon_1}{\sigma_1} << 1 \tag{2.116}$$

Joule power per unit area of the metallic surface

It is given by the inward flux of the Poynting vector into the metal surface:

$$P_1 = \mathrm{Re}\left\{\frac{1}{2}\left[\vec{E}_t \times \vec{H}_t^*\right]\cdot \vec{1}_z\right\}$$

Using (2.114) this becomes

$$P_1 = \mathrm{Re}\left\{\frac{1}{2}Z_s\left[\left(\vec{H}_t \times \vec{1}_z\right) \times \vec{H}_t^*\right]\cdot \vec{1}_z\right\} = \mathrm{Re}\left\{\frac{1}{2}Z_s\left|\vec{H}_t\right|^2\right\}$$

hence

$$\boxed{P_1 = \frac{1}{2}R_s\left|\vec{H}_t\right|^2} \tag{2.117}$$

where R_s [in Ω] is the surface resistance of the metal.

From (2.115) and (2.112) it reads

$$R_s = \sqrt{\frac{|\omega|\mu_1}{2\sigma_1}} = \frac{1}{\sigma_1\delta}$$

The last expression is the resistance of a metal sheet with thickness δ.

For copper at 20 °C,

$$R_s = 0.2609\ \ \mathrm{m}\Omega\left[f_{\mathrm{MHz}}\right]^{1/2} = 3.690\ \ \mathrm{m}\Omega\ \ \text{at}\ \ 200\ \ \mathrm{MHz}$$
$$= 8.250\ \ \mathrm{m}\Omega\ \ \text{at}\ \ 1\ \mathrm{GHz}$$

2.5 Applications of the complex Poynting vector

In isotropic media where $\xi = \eta = 0$ and at frequencies where dispersion is negligible, (2.32) reduces to

$$-\text{div}\left[\vec{E}\times\vec{H}^*\right] = E\cdot\vec{J}^* - \alpha_1\left[\vec{E}\cdot\varepsilon^*\vec{E}^* + \vec{H}^*\cdot\mu\vec{H}\right] + j\omega_1\left[-\vec{E}\cdot\varepsilon^*\vec{E}^* + \vec{H}^*\cdot\mu\vec{H}\right]$$

Remembering (2.6) and (2.11): $\varepsilon^* = \varepsilon' + j\varepsilon''$, $\mu = \mu' - j\mu''$, this relation can be rearranged as

$$-\text{div}\ \frac{1}{2}\left[\vec{E}\times\vec{H}^*\right] = \frac{1}{2}\vec{E}\cdot\vec{J}^* + \frac{\omega_1}{2}\left[\varepsilon''\left|\vec{E}\right|^2 + \mu''\left|\vec{H}\right|^2\right] - \frac{\alpha_1}{2}\left[\varepsilon'\left|\vec{E}\right|^2 + \mu'\left|\vec{H}\right|^2\right]$$

$$+\ j\frac{\omega_1}{2}\left[\left(\mu' + \frac{\alpha_1}{\omega_1}\mu''\right)\left|\vec{H}\right|^2 - \left(\varepsilon' + \frac{\alpha_1}{\omega_1}\varepsilon''\right)\left|\vec{E}\right|^2\right] \qquad (2.118)$$

When dispersion is negligible, the frequency is far from regions of anomalous dispersion; therefore

$$|\mu''| << \mu' \qquad |\varepsilon''| << \varepsilon'$$

Moreover, we shall always assume that $\alpha_1 << |\omega_1|$, so that the last term in (2.118) is well approximated by

$$j\frac{\omega_1}{2}\left(\mu'\left|\vec{H}\right|^2 - \varepsilon'\left|\vec{E}\right|^2\right)$$

Using (2.36) and (2.37), the integration of (2.118) over a volume V bounded by a closed surface S yields

$$\underbrace{-\oint\frac{1}{2}\left[\vec{E}\times\vec{H}^*\right]d\vec{S}}_{\text{Inward flux of the complex Poynting vector}} = \underbrace{\int\frac{1}{2}\vec{E}\cdot\vec{J}^*\,dV}_{\substack{\text{Complex power}\\ \text{absorbed by } \vec{J}}} + P_{\text{hysteresis}} - 2\alpha_1\left(W_e + W_m\right) + 2j\omega_1\left(W_m - W_e\right) \qquad (2.119)$$

In the presence of a current generator, in (2.119)

$$\vec{J} = \sigma\vec{E} + \vec{J}^e \qquad (2.120)$$

where $\sigma\vec{E}$ is a conduction current and $\vec{J}^e$ is an impressed current which, when oppositely directed to $\vec{E}$, represents a power generator. Then

$$\int\frac{1}{2}\vec{E}\cdot\vec{J}^*\,dV = \int\frac{1}{2}\sigma\left|\vec{E}\right|^2 dV + \int\frac{1}{2}\vec{E}\cdot\vec{J}^{e*}\,dV = P + \int\frac{1}{2}\vec{E}\cdot\vec{J}^{e*}\,dV \qquad (2.121)$$

P is the Joule power dissipated in V; the opposite of the second term is the complex power produced in V by impressed currents.

Example 1: Cavity excited by a transmission line, with no impressed currents in the cavity.

If the surface S contains a cross-section of the transmission line and surrounds the cavity walls at a distance of several skin depths, the inward flux of the complex Poynting vector differs from zero only in the cross-section of the line, and (2.119) becomes

$$\frac{1}{2}VI^* = P + 2j\omega\left(W_m - W_e\right) \qquad (2.122)$$

where P includes the power dissipated inside the cavity (by conduction and by hysteresis) and in the cavity walls; we have taken $\alpha_1 = 0$ because the RF generator maintains the field level constant.

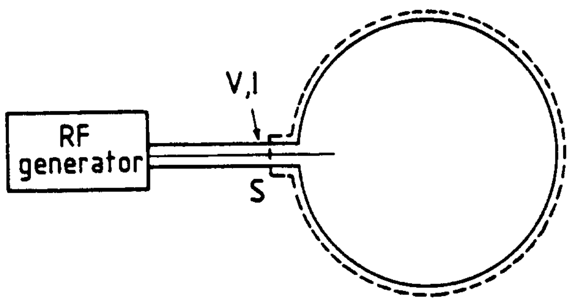


Fig. 2.5 A transmission line coupled to a cavity

The input impedance seen at S is

$$R + jX = \frac{V}{I} = \frac{2P}{|I|^2} + 4j\omega \frac{W_m - W_e}{|I|^2} \tag{2.123}$$

whereas the input admittance is

$$G + jB = \frac{I}{V} = \frac{2P}{|V|^2} - 4j\omega \frac{W_m - W_e}{|V|^2} \tag{2.124}$$

Equations (2.123) and (2.124) give an interpretation of the reactance X or the susceptance B in terms of $(W_m - W_e)$.

Example 2: Closed metallic cavity. Imaginary frequency shift due to power losses

If we now consider the cavity to be completely closed, the inward flux of the complex Poynting vector is zero in the metal, at some distance from the cavity walls. Fields can be maintained inside the cavity by impressed volume currents $\underline{J}^e$; this is the basis for excitation of cavities by electric currents (see Section 6). Without impressed volume currents in the cavity the fields are damped, and (2.119) becomes

$$0 = P - 2\alpha_1(W_e + W_m) + 2j\omega_1(W_m - W_e) \tag{2.125}$$

Equating real and imaginary parts, we obtain

$$\boxed{2\alpha_1 = \frac{P}{W}} \quad \text{and} \quad \boxed{W_m = W_e = \frac{1}{2}(W_m + W_e)} \tag{2.126}$$

where P is the total power dissipated inside the cavity and in the cavity walls; $W = W_e + W_m$ is the total electromagnetic energy stored inside the cavity and in the cavity walls.

This situation occurs when the fields oscillate at well defined eigenfrequencies, which are the resonances of the cavity. The ratio P/W is independent of the field level. It defines the *quality factor Q* of the cavity at a particular resonance through the relation

$$\frac{|\omega_1|}{Q} = \frac{P}{W} \tag{2.127}$$

so that

$$\boxed{\alpha_1 = \frac{|\omega_1|}{2Q}} \tag{2.128}$$

The complex frequency (2.27) reads

$$\boxed{\omega = \omega_1 + j\alpha_1 = \omega_1\left(1 + \frac{j\ \mathrm{sgn}(\omega_1)}{2Q}\right) \quad \text{or} \quad \omega^2 \approx \omega_1^2\left(1 + \frac{j\ \mathrm{sgn}(\omega_1)}{Q}\right)} \tag{2.129}$$

Let us now apply (2.119) by taking the surface S to be the inside surface of the cavity walls:

$$-\oint_{\text{cavity surface}} \frac{1}{2}\left[\vec{E} \times \vec{H}^*\right] \cdot \mathrm{d}\vec{S} = P_{\text{cavity}} - 2\alpha_1 (W_e + W_m)_{\text{cavity}} + 2j\omega_1 (W_m - W_e)_{\text{cavity}} \tag{2.130}$$

where P_{cavity} is the power dissipated inside the cavity.

Applying the boundary condition (2.114) at the surface of the cavity we obtain

$$\left[\vec{E} \times \vec{H}^*\right] \cdot \vec{1}_n = \left[\vec{E}_t \times \vec{H}_t^*\right] \cdot \vec{1}_n = Z_s \left|\vec{H}_t\right|^2$$

where Z_s is the surface impedance of the metal; from (2.115)

$$Z_s \approx \left[1 + j\ \mathrm{sgn}(\omega_1)\right] R_s \ .$$

Therefore the flux of the complex Poynting vector going into the metallic walls is

$$\oint_{\text{cavity surface}} \frac{1}{2}\left[\vec{E} \times \vec{H}^*\right] \cdot \mathrm{d}\vec{S} = \oint_{\text{cavity surface}} \frac{1}{2} Z_s \left|\vec{H}_t\right|^2 \mathrm{d}S \approx \left[1 + j\ \mathrm{sgn}(\omega_1)\right] \oint_{\text{cavity surface}} \frac{1}{2} R_s \left|\vec{H}_t\right|^2 \mathrm{d}S$$

$$= \left[1 + j\ \mathrm{sgn}(\omega_1)\right] \cdot P_{\text{wall}} \tag{2.131}$$

because the real part of this flux is the power dissipated in the cavity walls. We may now rewrite (2.130) as

$$0 \approx \left[1 + j\ \mathrm{sgn}(\omega_1)\right] P_{\text{wall}} + P_{\text{cavity}} - 2\alpha_1 (W_e + W_m)_{\text{cavity}} + 2j\omega_1 (W_m - W_e)_{\text{cavity}}$$

Subtracting from (2.125) yields

$$0 \approx -j\ \mathrm{sgn}(\omega_1) P_{\text{wall}} - 2\alpha_1 (W_e + W_m)_{\text{wall}} + 2j\omega_1 (W_m - W_e)_{\text{wall}} \tag{2.132}$$

The imaginary part yields

$$P_{\text{wall}} = 2|\omega_1| (W_m - W_e)_{\text{wall}} \tag{2.133}$$

In this relation the term W_e is not significant because the derivation of (2.132) is based on the approximate surface impedance (2.115), which implies that in a metal, as shown by (2.116), W_e is neglected with respect to W_m.

The real term in (2.132) can thus as well be written as $2\alpha_1(W_m - W_e)_{\text{wall}}$. Using (2.133) and (2.128) this reads

$$\frac{\alpha_1}{|\omega_1|} P_{\text{wall}} = \frac{1}{2Q} P_{\text{wall}} \ .$$

It is thus very small with respect to the first term in (2.132) and should be neglected.

Real frequency shift due to the penetration of the fields in the cavity walls

By analogy with (2.127), we define quality factors Q_w, Q_d related to the losses in the walls and inside the cavity by

$$\frac{|\omega_1|}{Q_w} = \frac{P_{\text{wall}}}{W_{\text{total}}} \quad \text{and} \quad \frac{|\omega_1|}{Q_d} = \frac{P_{\text{cavity}}}{W_{\text{total}}} \tag{2.134}$$

so that

$$\frac{1}{Q} = \frac{1}{Q_w} + \frac{1}{Q_d} \tag{2.135}$$

When the losses inside the cavity are due to the conductivity σ and the hysteresis of a uniform dielectric with permittivity ε, using (2.121), (2.37), (2.36) and (2.126) we simply have

$$\frac{1}{Q_d} = \frac{\int \left(\frac{1}{2}\sigma |\vec{E}|^2 + \frac{\omega_1}{2}\varepsilon'' |\vec{E}|^2 \right) \mathrm{d}V}{|\omega_1| \int \frac{1}{2}\varepsilon' |\vec{E}|^2 \mathrm{d}V} = \frac{\sigma + \omega_1 \varepsilon''}{|\omega_1| \varepsilon'} \tag{2.136}$$

From (2.126) we have taken $W_{\text{total}} = 2W_e$.

In the expression (2.134) of $1/Q_w$, P_{wall} is the real part of the flux (2.131) of the complex Poynting vector going into the metallic walls; it is proportional to the surface resistance R_s. On the other hand, the boundary condition (2.114) involves the surface impedance Z_s, which entails an imaginary part (i.e. a reactive part) in the flux (2.131). A complete treatment ([12], p. 360) shows that in (2.135), the term $1/Q_w$ is proportional to Z_s, which means that it should be multiplied by $[1 + j \operatorname{sgn}(\omega_1)]$. In (2.129), $1/Q$ should thus be replaced by

$$\frac{1}{Q_{\text{complex}}} = \frac{1 + j \operatorname{sgn}(\omega_1)}{Q_w} + \frac{1}{Q_d} \tag{2.137}$$

yielding

$$\omega^2 = \omega_0^2 \left[1 + \frac{j \operatorname{sgn}(\omega_1) - 1}{Q_w} + \frac{j \operatorname{sgn}(\omega_1)}{Q_d} \right] \tag{2.138}$$

Since the reactive part of Z_s produces a real frequency shift, we had to replace ω_1 in (2.129) by some ω_0. From (2.138) ω_0 is the resonant frequency of the same cavity with perfectly conducting walls and lossless dielectric; as such it is a real quantity. Comparing (2.138) to (2.129) we see that

$$\omega_1^2 \approx \omega_0^2 \left[1 - \frac{1}{Q_w} \right] \approx \omega_0^2 \left[1 + \frac{1}{Q_w} \right]^{-1} \tag{2.139}$$

The penetration of the fields into the cavity walls due to the skin effect always reduces the resonant frequency computed for the cavity with perfectly conducting walls. As shown by (2.139) the correct frequency shift is given by Q_w ; it will *not* be obtained correctly be assuming that the walls of the cavity recede uniformly in all directions by a distance δ.

Relation between the actual complex frequency and the frequency of the cavity with perfectly conducting walls and lossless dielectric

It is obtained from (2.138), which we write as

$$\boxed{\omega^2\left(1+\frac{1-j\ \mathrm{sgn}\,\mathrm{Re}(\omega)}{Q_w}-\frac{j\ \mathrm{sgn}\,\mathrm{Re}(\omega)}{Q_d}\right)=\omega_0^2} \tag{2.140}$$

With (2.135) this can also be rewritten as

$$\omega^2\left(1+\frac{1}{Q_w}-\frac{j\ \mathrm{sgn}\,\mathrm{Re}(\omega)}{Q}\right)=\omega_0^2 \tag{2.141}$$

Let us remember that ω_0 is the real frequency of the cavity with perfectly conducting walls and lossless dielectric. The correction factor in (2.140) will appear again in the theory of lossy waveguides and lossy resonant cavities (see Eq. (6.67) in section 6).

3. WAVEGUIDES

3.1 Monochromatic waves along cylindrical conductors with arbitrary cross section

The conductors are parallel to Oz. All media are supposed to be reciprocal, isotropic and uniform in space. Helmholtz's equations read (with $\vec{J} = \sigma\vec{E}, \quad \rho = 0$):

$$\begin{cases} \Delta\vec{E} + k^2\vec{E} = 0 & \text{with} \quad \text{div } \vec{E} = 0 \\ \Delta\vec{H} + k^2\vec{H} = 0 & \text{with} \quad \text{div } \vec{H} = 0 \end{cases} \tag{3.1}$$

where

$$\Delta \equiv \text{grad div} - \text{curl curl}$$

$$\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2} \tag{3.2}$$

$$k^2 = -j\omega\mu(\sigma + j\omega\varepsilon) \qquad \text{with } \mathrm{Im}(k) \le 0 \; . \tag{3.3}$$

If $\sigma = 0$,

$$k^2 = \omega^2\varepsilon\mu = \frac{\omega^2}{c^2}\varepsilon_r\mu_r \tag{3.4}$$

where

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} \, , \quad \mu_r = \frac{\mu}{\mu_0} \, .$$

Because of the form (3.2) of the Laplacian operator, Helmholtz's equations admit of solutions $e^{\pm\gamma z}$ in z.

By using Eq. (1.11) with z instead of t, the wave dependence on z can be represented as a linear superposition of complex exponential waves. In the following, we therefore consider only waves of the type

$$\vec{E}(x,y) \cdot e^{j\omega t - \gamma z} \quad \text{where} \quad \gamma = \alpha + j\beta \; , \quad \alpha \ge 0$$

and we factorize the complete exponential (in t and z) out of the equations.

Remark: When $\gamma = 0$, the two independent solutions $e^{-\gamma z}$ and $e^{\gamma z}$ coalesce into a single one (a constant); a second independent solution si then obtained by taking

$$\lim_{\gamma \to 0} -\frac{1}{2\gamma}\left(e^{-\gamma z} - e^{\gamma z}\right) = z \; .$$

For E_z, H_z Helmholtz's equations reduce to:

$$\begin{cases} \Delta_\perp E_z + k_c^2 E_z = 0 \\ \\ \Delta_\perp H_z + k_c^2 H_z = 0 \end{cases} \qquad \text{where} \quad k_c^2 = k^2 + \gamma^2 \; . \tag{3.5}$$

3.2 Equations for the transverse components of the fields

With $\vec{1}_z$ representing a unit vector along Oz, first observe that

$$\mathrm{curl}_x\vec{E}_\perp = -\frac{\partial E_y}{\partial z} = \gamma E_y \qquad \mathrm{curl}_y\vec{E}_\perp = \frac{\partial E_x}{\partial z} = -\gamma E_x$$

hence

$$\mathrm{curl}_\perp\left(\vec{E}_\perp\right) = \left[\gamma\vec{E}_\perp \times \vec{1}_z\right] \tag{3.6}$$

whereas

$$\mathrm{curl}\left(E_z\vec{1}_z\right) = \left[\mathrm{grad}\ E_z \times \vec{1}_z\right] = \left[\mathrm{grad}_\perp E_z \times \vec{1}_z\right]$$

or

$$\mathrm{curl}_\perp\left(E_z\vec{1}_z\right) = \left[\mathrm{grad}_\perp E_z \times \vec{1}_z\right]. \tag{3.7}$$

Similar relations apply for $\vec{H}$. Therefore,

$$\begin{aligned}
\mathrm{curl}_\perp\vec{E} &= \left[\left(\gamma\vec{E}_\perp + \mathrm{grad}_\perp E_z\right)\times\vec{1}_z\right] \quad \text{or} \quad \left[\vec{1}_z \times \mathrm{curl}_\perp\vec{E}\right] = \gamma\vec{E}_\perp + \mathrm{grad}_\perp E_z \\
\mathrm{curl}_\perp\vec{H} &= \left[\left(\gamma\vec{H}_\perp + \mathrm{grad}_\perp H_z\right)\times\vec{1}_z\right] \quad \text{or} \quad \left[\vec{1}_z \times \mathrm{curl}_\perp\vec{H}\right] = \gamma\vec{H}_\perp + \mathrm{grad}_\perp H_z\ .
\end{aligned} \tag{3.8}$$

From Maxwell's equations (2.1) and the constitutive relations (2.5), (2.10) we then obtain:

$$\boxed{\begin{aligned}
&\gamma\vec{E}_\perp - j\omega\mu\left[\vec{H}_\perp \times \vec{1}_z\right] &&= -\mathrm{grad}_\perp E_z \\
&-(\sigma + j\omega\varepsilon)\vec{E}_\perp + \gamma\left[\vec{H}_\perp \times \vec{1}_z\right] &&= -\left[\mathrm{grad}_\perp H_z \times \vec{1}_z\right]
\end{aligned}} \tag{3.9}$$

This system can be solved for the transverse components of the fields, yielding

$$\boxed{\begin{aligned}
\vec{E}_\perp &= -\frac{\gamma}{k_c^2}\mathrm{grad}_\perp E_z - \frac{j\omega\mu}{k_c^2}\left[\mathrm{grad}_\perp H_z \times \vec{1}_z\right] \\
\vec{H}_\perp &= \frac{\sigma + j\omega\varepsilon}{k_c^2}\left[\mathrm{grad}_\perp E_z \times \vec{1}_z\right] - \frac{\gamma}{k_c^2}\mathrm{grad}_\perp H_z
\end{aligned}} \tag{3.10}$$

With Maxwell's curl equations, the conditions $\mathrm{div}\ \vec{E} = 0$, $\mathrm{div}\ \vec{H} = 0$ are automatically satisfied. They read

$$\mathrm{div}\ \vec{E}_\perp = \gamma E_z\ , \qquad \mathrm{div}\ \vec{H}_\perp = \gamma H_z\ . \tag{3.11}$$

This can be verified at once from (3.10), by using (3.5) and the relations

$$\mathrm{div}\left[\mathrm{grad}_\perp E_z \times \vec{1}_z\right] = 0\ , \qquad \mathrm{div}\left[\mathrm{grad}_\perp H_z \times \vec{1}_z\right] = 0$$

which are direct consequences of (3.7).

3.3 Transverse deflecting force

The transverse deflecting force for particles with unit charge and velocity $\vec{v} = v\vec{1}_z$ is

$$\vec{F}_\perp = \vec{E}_\perp + \left[\vec{v} \times \vec{B}\right] = \vec{E}_\perp + \frac{v}{j\omega}\ j\omega\mu\left[\vec{1}_z \times \vec{H}_\perp\right].$$

Using the first equation of (3.9), this becomes

$$\vec{F}_\perp = \frac{v}{j\omega}\left[\left(\frac{j\omega}{v} - \gamma\right)\vec{E}_\perp - \mathrm{grad}_\perp E_z\right].$$

Putting

$$h = \omega / v \tag{3.12}$$

where v is considered to be independent of z, this equation may be written as

$$\vec{F}_\perp \mathrm{e}^{jhz} = \frac{v}{j\omega}\left[\frac{\partial}{\partial z}\left(\vec{E}_\perp \mathrm{e}^{jhz}\right) - \mathrm{grad}_\perp E_z \cdot \mathrm{e}^{jhz}\right] \tag{3.13}$$

in which form it no longer contains γ; therefore it is valid for any variation of fields along z.

When v depends on z, just replace e^{jhz} by $e^{j\omega\int dz/v}$. This phase factor accounts for the fact that the fields oscillate as $e^{j\omega t}$, where

$$t = t_0 + \int \frac{dz}{v}$$

is the time when the particle passes at position z.

3.3.1 Panofsky-Wenzel Theorem

$$\boxed{\underbrace{j\omega \int_0^{\ell} \vec{F}_\perp e^{jhz} \frac{dz}{v}}_{\substack{\text{Transverse momentum} \\ \text{gained by unit charge}}} = \underbrace{\left[\vec{E}_\perp e^{jhz}\right]_0^{\ell}}_{\substack{\text{vanishes when} \\ \vec{E}_\perp(0) = \vec{E}_\perp(\ell) = 0}} - \int_0^{\ell} \mathrm{grad}_\perp E_z \cdot \mathrm{e}^{jhz} dz} \tag{3.14}$$

This relation shows that when integrated between two positions where $\vec{E}_\perp$ is negligible, the transverse momentum transferred to the charge only depends on $\mathrm{grad}_\perp E_z$. In particular, there is no transverse kick to the particle when $E_z \equiv 0$.

The relation (3.14) applies at a given frequency. It describes a mathematical equivalence between the *total* transverse momentum gained by a unit charge and an integral over $\mathrm{grad}_\perp E_z$; but the actual deflecting force is given by (3.13). In particular, if (3.14) vanishes, the *total* transverse kick given by the electromagnetic field to the particle is zero; but at some z the deflecting force may be different from zero and can induce synchrotron radiation.

Example: Consider a relativistic particle passing between two parallel plates of length ℓ along z and infinitely wide along x.

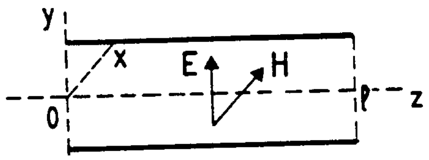


Fig. 3.1 A TEM wave between two parallel plates

The plates guide a TEM standing wave along z:

$$\begin{cases} E_y = E_0 \cos kz & h = \dfrac{\omega}{v} \approx \dfrac{\omega}{c} = k \\ H_x = j\dfrac{E_0}{\zeta_0}\sin kz & \zeta_0 \text{ is the impedance of free space .} \end{cases}$$

1) If we neglect the fringing fields at both ends of the plates, we are left with the first term of (3.14):

$$\left[\vec{E}_\perp e^{jhz}\right]_0^\ell = \vec{1}_y\left[E_y(\ell)\cdot e^{jh\ell} - E_y(0)\right] = E_0\vec{1}_y\left[\cos k\ell \cdot e^{jh\ell} - 1\right]$$

$$= E_0\vec{1}_y e^{jh\ell}\left[\cos k\ell - e^{-jh\ell}\right] \approx E_0\vec{1}_y e^{jh\ell}\cdot j\ \sin\ h\ell \quad \text{since } h \approx k\ .$$

2) If we integrate from $-\infty$ to $+\infty$ we are left with the second term of (3.14):

$$-\int_{-\infty}^{+\infty} \mathrm{grad}_\perp E_z \cdot e^{jhz} dz \neq 0$$

because of the fringing fields near $z = 0$ and $z = \ell$.

3.4 Classification of waves

Once the longitudinal fields E_z, H_z are known, the transverse fields can be obtained from (3.10). We must distinguish two cases.

a) *case* $k_c^2 = 0$

With (3.5), $\gamma = jk$, i.e. $\beta = k$ when $\sigma = 0$.

Such waves are synchronous with particles travelling along z with the velocity of light. For the transverse fields to stay finite in (3.10), we must have

$$\mathrm{grad}_\perp E_z = -\zeta\left[\mathrm{grad}_\perp H_z \times \vec{1}_z\right] \tag{3.15}$$

where

$$\boxed{\zeta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}\ , \quad \text{wave impedance of the medium}} \tag{3.16}$$

From (3.9),

$$\vec{E}_\perp = \zeta\left[\vec{H}_\perp \times \vec{1}_z\right] - \frac{1}{jk}\mathrm{grad}_\perp E_z \tag{3.17}$$

With (3.15) there are two possibilities:

1) $\mathrm{grad}_\perp E_z \neq 0$ with $\mathrm{grad}_\perp H_z \neq 0$

Since both E_z and H_z differ from zero, this is a hybrid *HE* or *EH* wave. Such waves are used to deflect ultra-relativistic particles in RF separators [19]. They describe the particular space-harmonic which is synchronous with the beam in the deflecting mode of a periodically loaded (e.g. disk-loaded) waveguide.

2) $\mathrm{grad}_\perp E_z = 0$ with $\mathrm{grad}_\perp H_z = 0$

If $k \neq 0$, from (3.17)

$$\vec{E}_\perp = \zeta\left[\vec{H}_\perp \times \vec{1}_z\right] \tag{3.18}$$

This is the case when $E_z = C_1$ with $H_z = C_2$.

Waves with $E_z = C_1 \neq 0, H_z = 0$ are used to accelerate ultra-relativistic particles. They describe the particular space-harmonic which is synchronous with the beam in the accelerating mode of a periodically loaded waveguide.

Fields with $E_z = 0, H_z = C_2 \neq 0$ can exist for $k = 0$ in metallic pipes. They are the solenoidal uniform magnetic field produced by azimuthal D.C. currents in the pipe walls.

Waves with $E_z = 0, H_z = 0$ are TEM (transverse electromagnetic) waves, satisfying (3.18). Since $E_z = 0$, a pure TEM wave does not accelerate nor deflect charged particles.

For TEM waves,
$$\mathrm{curl}_z\ \vec{E}_\perp = \mathrm{curl}_z \vec{E} = -j\omega\mu\ H_z = 0$$

and, from (3.11):
$$\mathrm{div}\ \vec{E}_\perp = \gamma\ E_z = 0\ . \tag{3.19a}$$

Similarly
$$\mathrm{curl}_z\ \vec{H}_\perp = \mathrm{curl}_z \vec{H} = (\sigma + j\omega\varepsilon)\ E_z = 0$$
$$\mathrm{div}\ \vec{H}_\perp = \gamma\ H_z = 0\ . \tag{3.19b}$$

b) case $k_c^2 \neq 0$

From (3.10) the transverse fields are obtained by superposition of two kinds of waves:

1. When $E_z = 0$: TE (transverse electric) or H (because $H_z \neq 0$) wave.

From (3.9),

$$\vec{E}_\perp = \frac{j\omega\mu}{\gamma}\left[\vec{H}_\perp \times \vec{1}_z\right] \qquad Z_{oH} = \frac{j\omega\mu}{\gamma}\text{(wave impedance)} \tag{3.20}$$

For H waves,
$$\mathrm{curl}_z \vec{E}_\perp = -j\omega\mu\ H_z \neq 0$$
$$\mathrm{div}\ \vec{E}_\perp = \gamma\ E_z = 0\ . \tag{3.21a}$$

Similarly
$$\mathrm{curl}_z \vec{H}_\perp = (\sigma + j\omega\varepsilon)\ E_z = 0$$
$$\mathrm{div}\ \vec{H}_\perp = \gamma\ H_z \neq 0\ . \tag{3.21b}$$

2. When $H_z = 0$: TM (transverse magnetic) or E (because $E_z \neq 0$) wave.

From (3.9),

$$\vec{E}_\perp = \frac{\gamma}{\sigma + j\omega\varepsilon}\left[\vec{H}_\perp \times \vec{1}_z\right] \qquad Z_{oE} = \frac{\gamma}{\sigma + j\omega\varepsilon} \tag{3.22}$$

For E waves,
$$\mathrm{curl}_z \vec{E}_\perp = -j\omega\mu\ H_z = 0$$
$$\mathrm{div}\ \vec{E}_\perp = \gamma\ E_z \neq 0\ . \tag{3.23a}$$

Similarly
$$\mathrm{curl}_z \vec{H}_\perp = (\sigma + j\omega\varepsilon)\ E_z \neq 0$$
$$\mathrm{div}\ \vec{H}_\perp = \gamma\ H_z = 0\ . \tag{3.23b}$$

Remark: The classification of waves according to the conditions (3.19), (3.21) or (3.23) in two dimensions is similar to the classification of modes as solenoidal or irrotational in a resonant cavity in three dimensions (see Section 6.2).

Hybrid waves

In a smooth waveguide containing a single homogeneous and isotropic dielectric, where both metal and dielectric are lossless, the most general field may be considered as a superposition of E and H waves; moreover these waves can be excited independently (see Section 3.6).

If one of the above conditions is not fulfilled, i.e. if

1) the waveguide is *not smooth*, but it is periodically loaded (see Section 5);

2) the cross-section of the waveguide contains *several isotropic* dielectrics;

3) the waveguide contains an *anisotropic* dielectric (as a ferrite);

4) either the metal or the dielectric is *lossy*;

then all six components of the electromagnetic field are present in the waveguide. This means that a wave is no longer E or H, but it is a mixture of E and H called *hybrid* wave.

The real guide can be reduced to the above ideal guide by continuous variation of one or more parameters (for example, the periodic loading can be made vanishingly small, the dielectric constant of a second dielectric present in the guide can be made very close to ε_0, ...). By such a transformation the original hybrid wave will be reduced to one of the E or H wave of the ideal guide. If, by continuous transformation, the hybrid wave is reduced to an H (or E) wave of the ideal guide, it is called an *HE* (or *EH*) wave; but this limiting procedure does not always yield consistent results ([19], p. 211).

There is one important exception to the above rule for hybrid waves. If the guide has rotational symmetry about its axis, i.e. if it is a circular guide, and if the fields have also rotational symmetry about the axis, then the waves are still pure E or pure H waves.

Since in a circular guide, the fields vary as $\cos m\varphi$ or $\sin m\varphi$, this means that waves with $m = 0$ are E or H waves; waves with $m \neq 0$ are *EH* or *HE* waves in any real guide.

The most important modes for beam dynamics are $m = 0$: longitudinal or accelerating modes; and $m = 1$: transverse or deflecting modes.

3.5 Surface impedance at the boundary of a medium

When the boundary conditions at the surface of a medium are expressed by

$$\vec{E}_t = Z_s\left[\vec{H}_t \times \vec{n}\right] \tag{3.24}$$

where $\vec{E}_t, \vec{H}_t$ are the field components tangent to the surface whilst $\vec{n}$ is a unit vector on the normal to the surface pointing towards the medium, then Z_s is called the *surface impedance* at the boundary of the medium.

<u>*Example*</u>: If the medium is a metal with conductivity σ,

$$Z_s = \zeta_{metal} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \approx [1 + j\ \mathrm{sgn}\,(\omega)]\sqrt{\frac{|\omega|\mu}{2\sigma}} = [1 + j\ \mathrm{sgn}\,(\omega)]R_s \tag{3.25}$$

where $R_s = 1/(\sigma\delta)$ is the surface resistance of the metal, δ being the skin depth.

It should be remembered that the boundary condition (3.24) on a metallic surface is based on the approximation

$$\frac{W_e}{W_m} \approx \frac{|\omega|\varepsilon}{\sigma} << 1 \qquad \text{in a metal}$$

which entails

$$|Z_s| << \sqrt{\frac{\mu}{\varepsilon}} \ .$$

The boundary condition (3.24) also assumes that the skin depth δ is small with respect to the thickness of the metallic surface; therefore it cannot be applied down to zero frequency.

3.6 Waveguides with $Z_s = 0$ (perfect conductors)

Boundary conditions

From (3.9) we have (see Fig. 3.3)

$$\gamma E_s = -j\omega\mu H_n - \frac{\partial E_z}{\partial s} \tag{3.26}$$

$$(\sigma + j\omega\varepsilon)E_s = -\gamma H_n - \frac{\partial H_z}{\partial n} \ . \tag{3.27}$$

On the waveguide walls, the tangential electric field must vanish so that

$$E_s = 0\ , \quad E_z = 0 \qquad \text{on the waveguide walls .} \tag{3.28}$$

Therefore, (3.26) requires that $\qquad \omega\mu\, H_n = 0,\ \text{ i.e. }\ H_n = 0 \qquad (3.29)$

as long as $\omega \neq 0$. When $\omega = 0$, the boundary conditions for $\vec{H}$ no longer take a simple form. In what follows we restrict our study to cases where (3.29) still applies; this is the case, for example, when the waveguide walls are superconductors. From (3.29) and (3.27) we then obtain the boundary conditions for the magnetic field:

$$H_n = 0\ , \qquad \frac{\partial H_z}{\partial n} = 0 \qquad \text{on the waveguide walls.} \tag{3.30}$$

3.6.1 Guides with external field

The cross section of the guide extends to infinity. It can be shown that the only possible waves are

TEM waves with
$$\sum_{n=1}^{N} I_n = 0\ .$$

This condition keeps the power flux of the wave finite. In particular, it prevents the propagation of waves along a single, perfectly conducting wire.

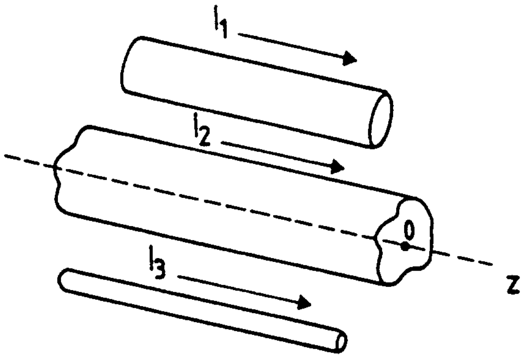


Fig. 3.2. N (=3) parallel, infinitely long conductors in open space

From (3.19a), $\quad \mathrm{curl}_z \vec{E}_\perp = 0 \quad \mathrm{div}\, \vec{E}_\perp = 0$

Therefore $\quad \vec{E}_\perp = -\mathrm{grad}_\perp V \quad \text{with} \quad \Delta_\perp V = 0$.

This reduces to an electrostatic problem in the transverse plane; the magnetic field can then be obtained from (3.18).

Examples: Lecher line ($N = 2$); polyphase lines ($N > 2$).

3.6.2 Guides with internal field

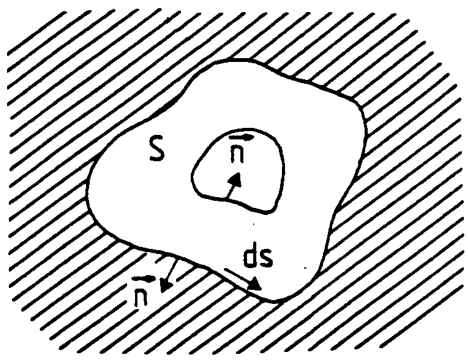


Fig. 3.3 Cross section of a waveguide with internal field

The cross section S of the guide is bounded by metallic walls of finite extent.

Differential equations, from (3.5):

$$\begin{aligned} &\text{for } H \text{ wave:} \quad \Delta_\perp H_z + k_c^2 H_z = 0 \\ &\text{for } E \text{ wave:} \quad \Delta_\perp E_z + k_c^2 E_z = 0 \end{aligned} \tag{3.31}$$

Boundary conditions

From (3.10),

$$E_s = -\frac{\gamma}{k_c^2}\frac{\partial E_z}{\partial s} + \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial n}$$

$$H_n = \frac{\sigma + j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial s} - \frac{\gamma}{k_c^2}\frac{\partial H_z}{\partial n}.$$

As long as $k_c^2 \neq 0$, the boundary conditions

$$E_z = 0 \ , \ \frac{\partial H_z}{\partial n} = 0 \qquad \text{on the perimeter } s \tag{3.32}$$

entail the other conditions $E_s = 0$, $H_n = 0$. When $k_c^2 = 0$, the full set of boundary conditions (3.28), (3.30) must be imposed.

Since the conditions (3.32) are different for E_z and H_z, the corresponding eigenvalues k_c^2 are different for E and H waves, which makes these waves independent solutions of Maxwell's equations. But it is important to notice that these waves can exist only in smooth, perfectly conducting pipes filled with a homogeneous isotropic dielectric; in all other cases* the waves are hybrid ($E_z \neq 0$ and $H_z \neq 0$).

Multiply the first equation (3.31) by H_z^* and integrate over S:

$$\underbrace{\oint_s H_z^* \frac{\partial H_z}{\partial n} ds}_{0} - \int_S |\text{grad}_\perp H_z|^2 dS + k_c^2 \int_S |H_z|^2 dS = 0 \tag{3.33}$$

hence k_c^2 is real ≥ 0. A similar equation applies for E_z.

Case $k_c^2 = 0$

From (3.33), $k_c^2 = 0$ implies $\text{grad}_\perp H_z = 0$ and $\text{grad}_\perp E_z = 0$. Therefore $H_z = C$ in case of an H wave; because of the boundary conditions (3.32): $E_z \equiv 0$, which means that $k_c^2 = 0$ is not an eigenvalue for E waves.

Since $E_z = 0$, the first equation of (3.21b) entails

$$\text{curl}_z \ \vec{H}_\perp = 0 \qquad \text{hence} \qquad \vec{H}_\perp = -\text{grad}_\perp \psi$$

where the scalar potential ψ can be multivalued if the cross section of the waveguide is multiply connected; the boundary condition (3.29) reads

$$\frac{\partial \psi}{\partial n} = 0 \qquad \text{on } s \ . \tag{3.34}$$

The second equation of (3.21b) entails

$$\Delta_\perp \psi + \gamma H_z = 0 \qquad \text{in } S \ .$$

Therefore, since H_z is a constant in S,

$$\oint_s \frac{\partial \psi}{\partial n} ds + \gamma H_z \int_S dS = 0 \ .$$

Hence:
$$\gamma H_z = 0 \qquad \text{and} \qquad \Delta_\perp \psi = 0 \qquad \text{in } S \ . \tag{3.35}$$

* Except for waves with no azimuthal variation in a circular pipe with finite conductivity; such waves are pure E or H waves.

If $H_z \neq 0$, the first condition entails $\gamma = 0$ hence $k = 0$. By analogy with (3.33), the second equation combined with (3.34) entails, if the domain S is simply connected:

$$\mathrm{grad}_\perp \psi = 0 \quad \text{or} \quad \vec{H}_\perp = 0 \ .$$

Let us now suppose that $k \neq 0$ (i.e. $\omega \neq 0$). Then, with (3.5), $\gamma \neq 0$ and with (3.35),

$$H_z \equiv 0 \ .$$

Therefore the wave is TEM. From (3.19a), $\mathrm{curl}_z \vec{E}_\perp = 0$ and $\mathrm{div}\ \vec{E}_\perp = 0$

hence
$$\vec{E}_\perp = -\mathrm{grad}_\perp V \quad \text{with} \quad V_i = \text{constant on } s_i \tag{3.36}$$

and
$$\Delta_\perp V = 0 \quad \text{in} \quad S \ . \tag{3.37}$$

The s_i represent the disconnected parts of the border of S. Equation (3.37) implies

$$0 = \int_S \Delta_\perp V \ dS = \sum_i \oint_{s_i} \frac{\partial V}{\partial n} \ ds \quad \text{or} \quad \sum_i q_i = 0$$

where q_i represents the charge that the part s_i carries per unit length along z.

Equation (3.37) also implies

$$0 = \int_S V^* \Delta_\perp V \ dS = \sum_i V_i^* \oint_{s_i} \frac{\partial V}{\partial n} \ ds \ - \int_S |\mathrm{grad}_\perp V|^2 \ dS$$

which entails, if the domain S is simply connected:

$$\mathrm{grad}_\perp V = 0 \quad \text{or} \quad \vec{E}_\perp = 0 \ .$$

Therefore a TEM wave cannot exist in a domain S which is simply connected. When the domain is multiply connected, TEM waves exist because V can take different values V_i on the different conductors.

Once $\vec{E}_\perp$ has been determined by (3.36) and (3.37), $\vec{H}_\perp$ is obtained from (3.18) as

$$\vec{H}_\perp = -\frac{1}{\zeta}\left[\vec{E}_\perp \times \vec{1}_z\right] . \tag{3.38}$$

Classification of eigenvalues

1) $H_z = C \neq 0$ is only possible if $k_c^2 = 0$ and $\omega = 0$; this case corresponds to a D.C. uniform magnetic field in the waveguide, which behaves as a solenoid. The fact that ω must be zero is also obvious from the boundary condition

$$0 = \oint E_s \ ds = -j\omega\mu \int H_z \ dS \ .$$

2) When S is simply connected (hollow guide), there is a countable infinite set of k_c^2 values starting from 0 (with $H_z = C$) for H waves and from a positive value for E waves. Moreover ([20], p. 292),

$$\text{smallest positive } k_c^2 \text{ for } H \text{ waves} < \text{smallest } k_c^2 \text{ for } E \text{ waves} \ .$$

The corresponding eigenfunctions H_z or E_z form a complete orthogonal system of scalar functions in S.

3) When S is multiply connected (cable with N conductors), $k_c^2 = 0$ is also possible with $(N - 1)$ independent TEM waves. The transverse electric field $E_\perp$ is the same as in the $(N - 1)$ independent electrostatic distributions of charges on the N conductors.

Interpretation of the eigenvalues k_c^2

In all cases, from (3.5), $$-\gamma^2 = k^2 - k_c^2.$$

If
$$\begin{array}{lll} k < k_c, & \gamma = \alpha & \text{evanescent wave} \\ k > k_c & \gamma = j\beta & \text{propagating wave} \\ k = k_c & \gamma = 0 & \text{cut - off frequency} \end{array}$$

Density of eigenvalues (Weyl's theorem)

The number $N(k_c^2)$ of modes which have a cut-off frequency smaller than k_c can be evaluated by an asymptotic formula valid for large k_c. This formula was first derived by H. Weyl ([21], p. 59).

Let S be the area, and ℓ be the perimeter of the waveguide cross-section. It can be shown that the distribution of eigenvalues k_c^2 of the scalar Helmholtz equation (3.31) combined with the boundary conditions (3.32) satisfies the following relations:

for H modes: $$N\left(k_c^2\right) \approx \frac{k_c^2 S}{4\pi} + \frac{k_c \ell}{4\pi}$$

for E modes: $$N\left(k_c^2\right) \approx \frac{k_c^2 S}{4\pi} - \frac{k_c \ell}{4\pi}$$

3.7 Waveguides with $Z_S \neq 0$

They may guide waves having an exponential decay along the normal external to the surface of the guide. These waves are called *surface waves* ([15], p. 697; [20], p. 377).

Examples: Sommerfeld wave on a single resistive wire (E wave); wave along a dielectric coating; wave outside an optical fibre.

3.8 Dispersion in waveguides

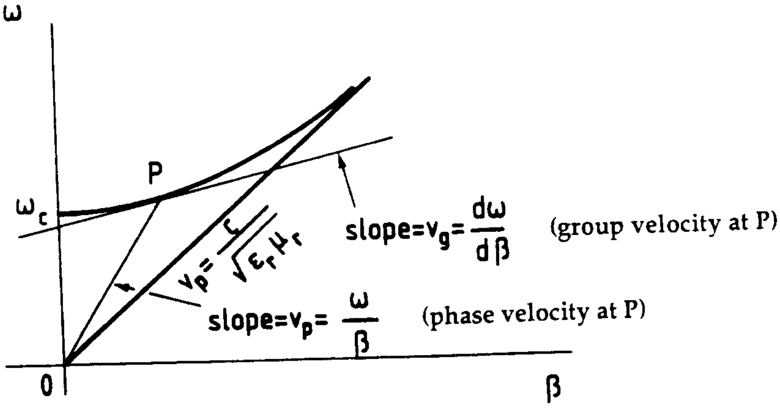


Fig. 3.4 Dispersion diagram for a waveguide

For a lossless waveguide, the dispersion relation is given by (3.5):

$$k^2 = k_c^2 + \beta^2 \ . \tag{3.39}$$

If the dielectric has zero conductivity, (3.4) applies and

$$\upsilon_p \upsilon_g = \frac{\omega}{\beta}\frac{d\omega}{d\beta} = \frac{1}{\varepsilon\mu}\frac{k}{\beta}\frac{dk}{d\beta} = \frac{1}{\varepsilon\mu} \quad \text{where} \quad \begin{array}{l} \upsilon_p > c(\varepsilon_r\mu_r)^{-1/2} \\ \upsilon_g < c(\varepsilon_r\mu_r)^{-1/2} \end{array} \tag{3.40}$$

When $\upsilon_g < \upsilon_p$, the dispersion is called normal. If the dielectric has $\sigma \neq 0$, $\upsilon_g > \upsilon_p$ and the dispersion is anomalous.

The energy velocity υ_e is defined by $P = \upsilon_e W_1$ where P is the power flux of a wave travelling along the guide, and W_1 is the energy stored per unit length of the guide in the travelling wave.

For a lossless waveguide, we shall prove in (3.65) that

$$\upsilon_p \upsilon_e = \frac{1}{\varepsilon\mu} \tag{3.41}$$

hence from (3.40): $\upsilon_e = \upsilon_g$.

It should be emphasized that this equality is tied to the absence of losses.

For a guide with finite conductivity or with dielectric losses $\gamma = \alpha + j\beta$, $P = P_o\, e^{-2\alpha z}$ for a travelling wave, and P_1 is the power lost per unit length of the guide:

$$P_1 = -\frac{dP}{dz} = 2\alpha P = 2\alpha\upsilon_e W_1 \ .$$

As in (2.127), the quality factor $Q(\omega)$ of the guide in the passband is defined by:

$$\frac{|\omega|}{Q} = \frac{P_1}{W_1} \tag{3.42}$$

where we take ω as real. Therefore

$$2\alpha = \frac{|\omega|}{\upsilon_e Q} \ . \tag{3.43}$$

Dispersion relation

For a waveguide with perfectly conducting walls and lossless dielectric ($\sigma = 0$, $\varepsilon'' = 0$), the dispersion relation is simply (3.5):

$$\omega_o^2 \ \varepsilon\mu = k_{co}^2 - \gamma^2 \tag{3.44}$$

where k_{co}^2 is obtained from Helmholtz's equations combined with the boundary conditions (3.32).

In order to take wall losses and dielectric losses ($\sigma \neq 0$ or $\varepsilon'' \neq 0$) into account, assuming they are small, we simply replace in (3.44) the frequency ω_o of the lossless case by the expression (2.141); this yields the dispersion relation

$$\boxed{\omega^2\varepsilon\mu\left(1+\frac{1}{Q_w}-\frac{j\ \mathrm{sgn}\,(\omega)}{Q}\right)=k_{co}^2-\gamma^2 \quad \text{where} \quad \gamma=\alpha+j\beta\ .} \tag{3.45}$$

Equating the imaginary parts we obtain

$$2\alpha\beta=\omega^2\varepsilon\mu\frac{\mathrm{sgn}\,(\omega)}{Q} \tag{3.46}$$

Since $\beta = \omega/\upsilon_p$, the comparison of (3.46) with (3.43) yields in the passband:

$$\upsilon_e\upsilon_p=\frac{1}{\varepsilon\mu} \tag{3.47}$$

which is identical to (3.41) obtained for a lossless waveguide.

Remark: In the stopband, the dispersion relation (3.45) no longer applies if Q_w and Q are still defined by (3.42). Nevertheless, (3.45) still applies if (2.141) is used as a definition for Q_w and Q. Obviously, such a definition of Q is consistent with the classical definitions (2.127), (2.134) in the passband; it has the advantage of providing an analytic continuation of the classical definitions in the stopband [22], where it yields simpler expressions.

For example, from (3.3) and (2.20) a dielectric conductivity σ has the effect to replace $k^2=\omega^2\varepsilon\mu$ by

$$\omega^2\varepsilon'\mu\left(1-j\frac{\sigma+\omega\varepsilon''}{\omega\varepsilon'}\right)=\omega^2\varepsilon'\mu\left(1-\frac{j\mathrm{sgn}(\omega)}{Q_d}\right)$$

where the last expression follows from (2.140); therefore, with the new definition,

$$\frac{1}{Q_d}=\frac{\sigma+\omega\varepsilon''}{|\omega|\varepsilon'} \tag{3.48}$$

The definition (3.48) yields the same result as (2.136), which however used the equality $W_m = W_e$. The new definition of Q_d extends the validity of (2.136) to cases where $W_m \neq W_e$; such a simple relation (3.48) would not be obtained with the classical definition (2.134) in the stopband, because it would require $W_m = W_e$, which to first order is fulfilled in the passband but not in the stopband, as shown later by Eq. (3.62).

Similarly [22], one may consider that imperfectly conducting walls produce a complex factor $Q_{w\,\mathrm{complex}}$ such that

$$\omega^2\left(1-\frac{j\ \mathrm{sgn}\,(\omega)}{Q_{w\,complex}}-\frac{j\mathrm{sgn}\,(\omega)}{Q_d}\right)=\omega_0^2\ . \tag{3.49}$$

If we compare with (2.140) we obtain

$$\frac{1}{Q_{w\ \mathrm{complex}}}=\frac{.+j\ \mathrm{sgn}(\omega)}{Q_w}$$

which is natural because $1/Q_{w\ \mathrm{complex}}$ is proportional to the wall impedance $Z_s = [1 + j\ \mathrm{sgn}(\omega)]R_s$. In fact, the generalized definition of Q leads to ([22], p. 303)

$$\frac{|\omega|}{Q_w}=\frac{\oint R_s(\omega)\vec{H}_t(\gamma)\cdot\vec{H}_t(-\gamma)ds}{\int_S\mu\vec{H}(\gamma)\cdot\vec{H}(-\gamma)dS} \tag{3.50}$$

where $\vec{H}(\gamma)$, $\vec{H}(-\gamma)$ represent the magnetic fields of waves travelling in the positive and in the negative z–direction along the lossy waveguide. From Eqs. (4.8) and (4.10) it will be seen that

$$\text{if} \quad \vec{H}(\gamma) \sim \left[\vec{H}_\perp + H_z \vec{1}_z\right] e^{-\gamma z}, \quad \text{then} \quad \vec{H}(-\gamma) \sim \left[-\vec{H}_\perp + H_z \vec{1}_z\right] e^{\gamma z}$$

hence

$$\vec{H}(\gamma) \cdot \vec{H}(-\gamma) \sim \left[-\vec{H}_\perp^2 + H_z^2\right]$$

which should be used in (3.50).

A first approximation is obtained when in (3.50), $\vec{H}(\gamma)$, $\vec{H}(-\gamma)$ are replaced by the fields in a lossless waveguide; then within the passband $-\gamma = \gamma^*$ and one may take

$$\vec{H}(-\gamma) = C\vec{H}(\gamma)^* \quad \text{where } C \text{ is a constant,}$$

so that (3.50) then reduces to the classical definition (2.134).

3.9 Longitudinal and transverse energy densities

From (3.8) and Maxwell's equations (2.1) we derived (3.9), which we here rewrite in its complex conjugate form (assuming ε, μ, ω to be real quantities):

$$\begin{cases} \gamma^* \vec{E}_\perp^* + j\omega\mu\left[\vec{H}_\perp^* \times \vec{1}_z\right] = -\mathrm{grad}_\perp \vec{E}_z^* \\ -(\sigma - j\omega\varepsilon)\vec{E}_\perp^* + \gamma^*\left[\vec{H}_\perp^* \times \vec{1}_z\right] = -\left[\mathrm{grad}_\perp H_z^* \times \vec{1}_z\right] \end{cases} \tag{3.51}$$

By taking the alternate form of the equations in (3.8) one would obtain

$$\begin{cases} \gamma\left[\vec{E}_\perp \times \vec{1}_z\right] + j\omega\mu\vec{H}_\perp = -\left[\mathrm{grad}_\perp E_z \times \vec{1}_z\right] \\ (\sigma + j\omega\varepsilon)\left[\vec{E}_\perp \times \vec{1}_z\right] + \gamma\vec{H}_\perp = -\mathrm{grad}_\perp H_z \ . \end{cases} \tag{3.52}$$

Let us dot multiply (3.51) respectively by $\varepsilon\vec{E}_\perp$ and $\vec{E}_\perp$:

$$\gamma^*\varepsilon\left|\vec{E}_\perp\right|^2 + j\omega\varepsilon\mu \ \vec{1}_z \cdot \left[\vec{E}_\perp \times \vec{H}_\perp^*\right] = -\varepsilon\vec{E}_\perp \cdot \mathrm{grad}_\perp E_z^* = E_z^* \ \mathrm{div}\left(\varepsilon\vec{E}_\perp\right) - \mathrm{div}\left(E_z^*\varepsilon\vec{E}_\perp\right)$$

$$-(\sigma - j\omega\varepsilon)\left|\vec{E}_\perp\right|^2 + \gamma^*\vec{1}_z \cdot \left[\vec{E}_\perp \times \vec{H}_\perp^*\right] = -\left[\vec{1}_z \times \vec{E}_\perp\right] \cdot \mathrm{grad}_\perp H_z^* = H_z^* \ \mathrm{div}\left(\vec{1}_z \times \vec{E}_\perp\right) - \mathrm{div}\left(H_z^*\left[\vec{1}_z \times \vec{E}_\perp\right]\right)$$

At this point we assume that ε, μ do not vary in space. Then, from (3.11),

$$\mathrm{div}\left(\varepsilon\vec{E}_\perp\right) = \gamma\varepsilon E_z$$

whereas $\quad \mathrm{div}\left(\vec{1}_z \times \vec{E}_\perp\right) = -\vec{1}_z \cdot \mathrm{curl}\ \vec{E}_\perp = -\mathrm{curl}_z\ \vec{E} = j\omega\mu\ H_z$.

Therefore the previous equations become

$$\begin{cases} \gamma^*\varepsilon\left|\vec{E}_\perp\right|^2 - \gamma\varepsilon\left|E_z\right|^2 = -j\omega\varepsilon\mu\ \left[\vec{E} \times \vec{H}^*\right] \cdot \vec{1}_z - \mathrm{div}\left(E_z^*\varepsilon\vec{E}_\perp\right) \\ (\sigma - j\omega\varepsilon)\left|\vec{E}_\perp\right|^2 + j\omega\mu\left|H_z\right|^2 = \gamma^*\left[\vec{E} \times \vec{H}^*\right] \cdot \vec{1}_z + \mathrm{div}\left(\vec{H}_z^*\left[\vec{1}_z \times \vec{E}_\perp\right]\right) . \end{cases} \tag{3.53}$$

In a similar manner we dot multiply (3.52) respectively by $\vec{H}_\perp^*$ and $\mu\vec{H}_\perp^*$:

$$-\gamma \vec{1}_z \cdot \left[\vec{E}_\perp \times \vec{H}_\perp^*\right] + j\omega\mu \ \left|\vec{H}_\perp\right|^2 = -\left[\vec{1}_z \times \vec{H}_\perp^*\right] \cdot \mathrm{grad}_\perp E_z = E_z \ \ \mathrm{div}\left[\vec{1}_z \times \vec{H}_\perp^*\right] - \mathrm{div}\left(E_z\left[\vec{1}_z \times \vec{H}_\perp^*\right]\right)$$

$$-(\sigma + j\omega\varepsilon)\mu \vec{1}_z \cdot \left[\vec{E}_\perp \times \vec{H}_\perp^*\right] + \gamma\mu\left|\vec{H}_\perp\right|^2 = -\mu\vec{H}_\perp^* \cdot \mathrm{grad}_\perp H_z = H_z \ \ \mathrm{div}\left(\mu\vec{H}_\perp^*\right) - \mathrm{div}\left(H_z \mu\vec{H}_\perp^*\right)$$

But $\quad \mathrm{div}\left[\vec{1}_z \times \vec{H}_\perp^*\right] = -\vec{1}_z \cdot \mathrm{curl} \ \vec{H}_\perp^* = -\mathrm{curl}_z \ \vec{H}^* = -(\sigma - j\omega\varepsilon)\vec{E}_z^*$

and from (3.11), div $(\mu\vec{H}_\perp^*) = \gamma^*\mu H_z^*$ so that the previous equations become

$$\begin{cases} j\omega\mu\left|\vec{H}_\perp\right|^2 + (\sigma - j\omega\varepsilon)\left|E_z\right|^2 = \gamma\left[\vec{E} \times \vec{H}^*\right] \cdot \vec{1}_z - \mathrm{div}\left(E_z\left[\vec{1}_z \times \vec{H}_\perp^*\right]\right) \\ \gamma\mu\left|\vec{H}_\perp\right|^2 - \gamma^*\mu\left|H_z\right|^2 = (\sigma + j\omega\varepsilon)\mu\left[\vec{E} \times \vec{H}^*\right] \cdot \vec{1}_z - \mathrm{div}\left(H_z \mu\vec{H}_\perp^*\right) \end{cases} . \tag{3.54}$$

If dispersion, i.e. the variation of ε and μ with ω is neglected, following (2.36) the time averages of the transverse and longitudinal energy densities per unit length of the waveguide are defined by:

$$\begin{array}{ll} W_\perp^e = \frac{1}{4}\int \varepsilon\left|\vec{E}_\perp\right|^2 dS & W_\perp^m = \frac{1}{4}\int \mu\left|\vec{H}_\perp\right|^2 dS \\ W_z^e = \frac{1}{4}\int \varepsilon\left|E_z\right|^2 dS & W_z^m = \frac{1}{4}\int \mu\left|H_z\right|^2 dS \end{array} \tag{3.55}$$

Then

$$\begin{array}{ll} W^e = W_\perp^e + W_z^e & W^m = W_\perp^m + W_z^m \\ W_\perp = W_\perp^e + W_\perp^m & W_z = W_z^e + W_z^m \end{array}$$

$$W_\perp + W_z = W^e + W^m$$

The flux of the complex Poynting vector through the waveguide cross section is separated into real and imaginary parts as(*)

$$\int_S \frac{1}{2}\left[\vec{E} \times \vec{H}^*\right] \cdot d\vec{S} = P + jQ \ . \tag{3.56}$$

Integrating (3.53) and (3.54) over the waveguide cross section (with ε, μ, σ, assumed to be constant) yields (see Fig. 3.3):

$$2\gamma^* W_\perp^e - 2\gamma W_z^e = -j\omega\varepsilon\mu(P + jQ) \qquad -\frac{1}{2}\oint E_z^* \varepsilon \vec{E} \cdot \vec{1}_n ds \tag{3.57a}$$

$$2\left(\frac{\sigma}{\varepsilon} - j\omega\right)W_\perp^e + 2j\omega \ W_z^m = \gamma^*(P + jQ) \ -\frac{1}{2}\oint H_z^* \vec{E} \cdot \vec{1}_s ds \tag{3.57b}$$

$$2j\omega \ W_\perp^m + 2\left(\frac{\sigma}{\varepsilon} - j\omega\right)W_z^e = \gamma \ (P + jQ) \quad +\frac{1}{2}\oint E_z \vec{H}^* \cdot \vec{1}_s ds \tag{3.57c}$$

$$2\gamma \ W_\perp^m - 2\gamma^* W_z^m = (\sigma + j\omega\varepsilon)\mu(P + jQ) \qquad -\frac{1}{2}\oint H_z \mu\vec{H}^* \cdot \vec{1}_n ds \ . \tag{3.57d}$$

When the waveguide walls are perfect conductors, all the contour integrals in (3.57) vanish because of the boundary conditions

(*) In this section, Q is the imaginary part of the complex power flow along the waveguide. Hopefully, there should be no confusion with the rest of this chapter, where Q always represents a quality factor.

$$\left[\vec{n}\times\vec{E}\right]=0 \qquad \vec{n}\cdot\vec{H}=0 \ .$$

In this case, by separating real and imaginary parts in (3.57a), (3.57d), (3.57b), (3.57c) we obtain

$$\begin{aligned}
&2\alpha\left(W_\perp^e - W_z^e\right) = \omega\varepsilon\mu Q &\qquad& 2\beta\left(W_\perp^e + W_z^e\right) = \omega\varepsilon\mu P \\
&2\alpha\left(W_\perp^m - W_z^m\right) = \sigma\mu P - \omega\varepsilon\mu Q &\qquad& 2\beta\left(W_\perp^m + W_z^m\right) = \sigma\mu Q + \omega\varepsilon\mu P \\
&2\frac{\sigma}{\varepsilon}W_\perp^e = \alpha P + \beta Q &\qquad& 2\omega\left(W_\perp^e - W_z^m\right) = -\alpha Q + \beta P \\
&2\frac{\sigma}{\varepsilon}W_z^e = \alpha P - \beta Q &\qquad& 2\omega\left(W_\perp^m - W_z^e\right) = \alpha Q + \beta P \ .
\end{aligned} \tag{3.58}$$

This set of equations is equivalent to

$$\begin{aligned}
&2\alpha\left(W_\perp^e - W_z^e\right) = \omega\varepsilon\mu Q &\qquad& 2\beta\left(W_\perp^e + W_z^e\right) = \omega\varepsilon\mu P \\
&2\alpha\left(W_\perp - W_z\right) = \sigma\mu P &\qquad& 2\beta\left(W^m - W^e\right) = \sigma\mu Q \\
&\frac{\sigma}{\varepsilon}\left(W_\perp^e + W_z^e\right) = \alpha P &\qquad& \omega\left(W^m - W^e\right) = \alpha Q \\
&\frac{\sigma}{\varepsilon}\left(W_\perp^e - W_z^e\right) = \beta Q &\qquad& \omega\left(W_\perp - W_z\right) = \beta P
\end{aligned} \tag{3.59}$$

The eight equations have been linked together in four pairs. The pair which contains $\left(W_\perp^e + W_z^e\right) > 0$ implies that

$$2\alpha\beta = \omega\mu\sigma \ . \tag{3.60}$$

With this relation, each pair reduces to a single equation; therefore (3.59) is equivalent to (3.60) and the following four equations:

$$\begin{cases}
2\alpha\left(W_\perp^e - W_z^e\right) = \omega\varepsilon\mu Q & \qquad 2\beta \ W^e = \omega\varepsilon\mu P \\
\omega\left(W^m - W^e\right) = \alpha Q & \qquad \omega\left(W_\perp - W_z\right) = \beta P
\end{cases} \tag{3.61}$$

which entail

$$\boxed{P = \frac{\omega}{\beta}\left(W_\perp - W_z\right) = \frac{\beta}{\omega\varepsilon\mu}2W^e \qquad Q = \frac{\omega}{\alpha}\left(W^m - W^e\right) = \frac{\alpha}{\omega\varepsilon\mu}2\left(W_\perp^e - W_z^e\right) \ .} \tag{3.62}$$

Remembering (3.3) and (3.5), it should be noticed that (3.60) is equivalent to

$$\mathrm{Im}\left(k_c^2\right) = \mathrm{Im}\left(k^2 + \gamma^2\right) = 0 \tag{3.63}$$

which confirms that $\left(k_c^2\right)$ is real for a waveguide with perfectly conducting walls.

In the particular case where $\sigma = 0$, i.e. when the waveguide dielectric is also lossless, we have $\alpha\beta = 0$. There are two possibilities:

1) *Propagating waves: $\alpha = 0$ (passband)*

Then $Q = 0$ and $W^m = W^e = 1/2\ W$. The power flow across the waveguide is purely real; it is given by

$$P = \frac{\omega}{\beta}\left(W_\perp - W_z\right) = \upsilon_p\left(W_\perp - W_z\right) \quad \text{since } \upsilon_p = \frac{\omega}{\beta} \tag{3.64a}$$

or by

$$P = \frac{\beta}{\omega\varepsilon\mu} W = \upsilon_e W \quad \text{hence} \quad \upsilon_e = \frac{\beta}{\omega\varepsilon\mu} . \tag{3.64b}$$

It follows that

$$\upsilon_e \upsilon_p = \frac{1}{\varepsilon\mu} > 0 \quad \text{and} \quad 0 < \frac{\upsilon_e}{\upsilon_p} = \frac{W_\perp - W_z}{W_\perp + W_z} < 1 . \tag{3.65}$$

It thus appears that the difference between υ_e and υ_p is tied to the existence of longitudinal fields; in particular $\upsilon_e = \upsilon_p$ for a TEM wave only.

2) *Evanescent waves: $\beta = 0$ (stopband)*

Then $P = 0$ and $W_\perp = W_z$. The power flow across the waveguide is purely reactive; it is given by

$$Q = \frac{\omega}{\alpha}\left(W^m - W^e\right) \tag{3.66a}$$

or by

$$Q = \frac{2\alpha}{\omega\varepsilon\mu}\left(W_\perp^e - W_z^e\right) = -\frac{2\alpha}{\omega\varepsilon\mu}\left(W_\perp^m - W_z^m\right) . \tag{3.66b}$$

The expression (3.66a) can be obtained directly by applying (2.119), with $\vec{J} = 0$ and $\alpha_1 = 0$, to the volume enclosed between two cross sections of the waveguide at z and $(z + dz)$.

The relations (3.58) can be generalized to include the case of waveguides containing lossy and nonreciprocal media [23]. The relation (3.64a) is also valid for dielectric waveguides ([17], p. 43).

3.10 Attenuation of waveguides

We suppose that the attenuation is only due to resistive wall losses. In the passband where α is small, from (3.62) we may take $W_m \approx W_e$. Using (2.117) and (2.36) in (3.42) we obtain

$$\frac{|\omega|}{Q_w} = \frac{\frac{1}{2}\oint R_s \left|\vec{H}_t\right|^2 \mathrm{d}s}{\frac{1}{2}\int_S \mu \left|\vec{H}\right|^2 \mathrm{d}S} \tag{3.67}$$

where the upper integral is taken on the perimeter of the waveguide and the lower integral is taken over the cross-section (see Fig. 3.3). These integrals are computed for a lossles waveguide.

Using (3.4) and (3.5), the attenuation is then computed from (3.46) as

$$2\alpha = \frac{|\omega|}{Q_w}\frac{\omega\varepsilon\mu}{\beta} = \frac{|\omega|}{Q_w}\frac{|k|\sqrt{\varepsilon\mu}}{\sqrt{k^2 - k_c^2}} \tag{3.68}$$

This formula obviously breaks down when $k = k_c$; in this case one should compute β from (3.45) and not from (3.5), which only applies for a lossless waveguide.

H wave

By (3.10) we can express both integrals in (3.67) in terms of H_z.

$$\oint |\vec{H}_t|^2 \mathrm{d}s = \oint |\vec{H}_\perp|^2 \mathrm{d}s + \oint |H_z|^2 \mathrm{d}s = \left|-\frac{\gamma}{k_c^2}\right|^2 \oint \left|\frac{\partial H_z}{\partial s}\right|^2 \mathrm{d}s + \oint |H_z|^2 \mathrm{d}s \tag{3.69}$$

With (3.33) and (3.5):

$$\int_S |\vec{H}|^2 \mathrm{d}S = \left|-\frac{\gamma}{k_c^2}\right|^2 \int_S |\mathrm{grad}_\perp H_z|^2 \mathrm{d}S + \int_S |\vec{H}_z|^2 \mathrm{d}S = \left(\frac{|\gamma|^2}{k_c^2} + 1\right) \int_S |H_z|^2 \mathrm{d}S = \frac{k^2}{k_c^2} \int_S |H_z|^2 \mathrm{d}S \tag{3.70}$$

Therefore (3.67) becomes

$$\frac{|\omega|}{Q_w} = \frac{R_s}{\mu} \frac{k_c^2}{k^2} \frac{\left(\frac{k^2}{k_c^2} - 1\right) \oint \frac{1}{k_c^2} \left|\frac{\partial H_z}{\partial s}\right|^2 \mathrm{d}s + \oint |H_z|^2 \mathrm{d}s}{\int_S |H_z|^2 \mathrm{d}S}$$

or

$$\frac{|\omega|}{Q_w} = \frac{R_s}{\mu} \frac{\oint \frac{1}{k_c^2} \left|\frac{\partial H_z}{\partial s}\right|^2 \mathrm{d}s + \left[\oint |H_z|^2 \mathrm{d}s - \oint \frac{1}{k_c^2} \left|\frac{\partial H_z}{\partial s}\right|^2 \mathrm{d}s\right] \frac{k_c^2}{k^2}}{\int_S |H_z|^2 \mathrm{d}S}$$

This can be rewritten as

$$\frac{|\omega|}{Q_w} = \frac{R_s}{\mu} \left[A + B \frac{k_c^2}{k^2}\right] \tag{3.71}$$

where

$$A = \frac{\oint \frac{1}{k_c^2} \left|\frac{\partial H_z}{\partial s}\right|^2 \mathrm{d}s}{\int_S |H_z|^2 \mathrm{d}S}, \quad B = \frac{\oint |H_z|^2 \mathrm{d}s - \oint \frac{1}{k_c^2} \left|\frac{\partial H_z}{\partial s}\right|^2 \mathrm{d}s}{\int_S |H_z|^2 \mathrm{d}S} \tag{3.72}$$

are coefficients independent of frequency.

From (3.68) and (3.71) we have

$$2\alpha = \frac{R_s}{\zeta} \left(1 - \frac{k_c^2}{k^2}\right)^{-1/2} \left[A + B \frac{k_c^2}{k^2}\right] \sim \left|\frac{k}{k_c}\right|^{1/2} \left(1 - \frac{k_c^2}{k^2}\right)^{-1/2} \left[A + B \frac{k_c^2}{k^2}\right] \tag{3.73}$$

where we have used the $|\omega|^{1/2}$ dependence of R_s shown in (2.115). It is easy to see that as a function of frequency, α reaches a minimum when

$$\frac{k^2}{k_c^2} = \frac{3}{2}\left(\frac{B}{A} + 1\right) + \sqrt{\frac{9}{4}\left(\frac{B}{A} + 1\right)^2 - \frac{B}{A}} \tag{3.74}$$

From (3.72) we always have

$$\frac{B}{A} + 1 = \frac{\oint |H_z|^2 \mathrm{d}s}{\oint \frac{1}{k_c^2} \left|\frac{\partial H_z}{\partial s}\right|^2 \mathrm{d}s} > 0$$

Nevertheless, for all waveguide cross-sections where an explicit solution H_z can be found, we have the more stringent inequality $B/A > 0$. Therefore, since (3.74) is an ever increasing function of B/A, the minimum of attenuation is reached at a frequency

$$\frac{k^2}{k_c^2} > 3 \tag{3.75}$$

In particular, if $A = 0$ (which means $\partial H_z/\partial s \equiv 0$ on the waveguide periphery), the minimum of attenuation recedes to $k^2 = \infty$; in that case the attenuation is an ever decreasing function of frequency. This property makes very attractive the H modes with H_z = constant at a given z on the waveguide periphery; such modes exist as the rotationally symmetrical H_{0n} modes in circular and coaxial waveguides (see later).

E wave

By (3.10) we can express both integrals in (3.67) in terms of E_z.

$$\oint |\vec{H}_t|^2 \mathrm{d}s = \oint |\vec{H}_\perp|^2 \mathrm{d}s = \left|\frac{\sigma + j\omega\varepsilon}{k_c^2}\right|^2 \oint \left|\frac{\partial E_z}{\partial n}\right|^2 \mathrm{d}s \tag{3.76}$$

With (3.33):

$$\int_S |\vec{H}|^2 \mathrm{d}S = \int_S |\vec{H}_\perp|^2 \mathrm{d}S = \left|\frac{\sigma + j\omega\varepsilon}{k_c^2}\right|^2 \int_S |\mathrm{grad}_\perp E_z|^2 \mathrm{d}S = \left|\frac{\sigma + j\omega\varepsilon}{k_c^2}\right|^2 k_c^2 \int_S |E_z|^2 \mathrm{d}S \tag{3.77}$$

Therefore (3.67) becomes

$$\frac{|\omega|}{Q_w} = \frac{R_s}{\mu} A \qquad \text{where} \qquad A = \frac{\oint \frac{1}{k_c^2} \left|\frac{\partial E_z}{\partial n}\right|^2 \mathrm{d}s}{\int_S |E_z|^2 \mathrm{d}S} \tag{3.78}$$

A is a coefficient independent of frequency.

From (3.68) and (3.78) we have

$$2\alpha = \frac{R_s}{\zeta}\left(1 - \frac{k_c^2}{k^2}\right)^{-1/2} A \tag{3.79}$$

Comparing with (3.73), we immediately see from (3.74) where $B = 0$ that the attenuation α reaches a minimum when

$$\frac{k^2}{k_c^2} = 3 \tag{3.80}$$

3.11 Some waveguides of simple shape

3.11.1 Rectangular waveguide

When the cross section of the waveguide is a rectangle of sides a, b (with $a \geq b$), the Helmholtz equation is separable in the transverse coordinates x, y.

H waves

$$H_z = H_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \tag{3.81}$$

$$\boxed{k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad m,\ n = 0,\ 1,\ 2,\ \ldots}$$

The mode with the lowest cut-off frequency, apart from $H_z = C$ (for $m = n = 0$), is the H_{10} mode.

From (3.10) we obtain:

$$\begin{aligned} E_x &= H_{mn}\frac{j\omega\mu}{k_c^2}\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) & H_x &= H_{mn}\frac{\gamma}{k_c^2}\frac{m\pi}{a}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \\ E_y &= -H_{mn}\frac{j\omega\mu}{k_c^2}\frac{m\pi}{a}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) & H_y &= H_{mn}\frac{\gamma}{k_c^2}\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right). \end{aligned} \tag{3.82}$$

E waves

$$E_z = E_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \tag{3.83}$$

$$\boxed{k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad m,\ n = 1,\ 2,\ \ldots}$$

The mode with the lowest cut-off frequency is the E_{11} mode.

$$\begin{aligned} E_x &= -E_{mn}\frac{\gamma}{k_c^2}\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) & H_x &= E_{mn}\frac{\sigma + j\omega\varepsilon}{k_c^2}\frac{n\pi}{b}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \\ E_y &= -E_{mn}\frac{\gamma}{k_c^2}\frac{n\pi}{b}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) & H_y &= -E_{mn}\frac{\sigma + j\omega\varepsilon}{k_c^2}\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right). \end{aligned} \tag{3.84}$$

Remark: The degeneracy of cutoff frequencies for H and E waves in a rectangular waveguide is removed when wall losses are taken into account ([12], p. 352; [15], p. 350).

3.11.2 Circular waveguide

When the cross section of the waveguide is a circle of radius a, the Helmholtz equation is again separable in the transverse coordinates r, φ. In the following, $J_m(z)$ is the Bessel function of the first kind of order m and argument z.

H waves $\qquad H_z = H_{mn}J_m(k_c r)\cos m\varphi \quad$ where $\ k_c a = j'_{mn}$, a root of $J'_m(k_c a) = 0$ $\qquad$ (3.85)

The integer $n = 0, 1, ..$ counts the zeros of $J'_m(x)$; $n = 0$ corresponds to $k_c a = 0$ when $m = 0$, and to $k_c a > 0$ when $m \neq 0$.

$$\boxed{(k_c a)^2 \approx \left(n + \frac{2m+1}{4}\right)^2 \pi^2 - \frac{4m^2+3}{4} \qquad m = 0,\ 1,\ 2,\ \ldots \qquad n = 0,\ 1,\ 2,\ \ldots}$$

For historical reasons these modes are designated as $H_{m,n+1}$ when $m \neq 0$, although the notation H_{mn} would be more logical.

The mode with the lowest cut-off frequency, apart from $H_z = C$ (for $m = 0$, $n = 0$, $k_c = 0$), is the H_{11} mode.

$$\begin{aligned} E_r &= H_{mn}\frac{j\omega\mu}{k_c}\frac{m}{k_c r}J_m(k_c r)\sin m\varphi & H_r &= -H_{mn}\frac{\gamma}{k_c}J'_m(k_c r)\cos m\varphi \\ E_\varphi &= H_{mn}\frac{j\omega\mu}{k_c}J'_m(k_c r)\cos m\varphi & H_\varphi &= H_{mn}\frac{\gamma}{k_c}\frac{m}{k_c r}J_m(k_c r)\sin m\varphi \end{aligned} \tag{3.86}$$

E-waves $E_z = E_{mn} J_m(k_c r)\cos m\varphi$ where $k_c a = j_{mn}$, n^{th} positive root of $J_m(k_c a) = 0$ (3.87)

$$\boxed{(k_c a)^2 \approx \left(n + \frac{2m-1}{4}\right)^2 \pi^2 - \frac{4m^2-1}{4} \qquad m = 0,\ 1,\ 2,\ \ldots \qquad n = 1,\ 2,\ \ldots}$$

The mode with the lowest cut-off frequency is the E_{01} mode.

$$\begin{aligned} E_r &= -E_{mn} \frac{\gamma}{k_c} J_m'(k_c r)\cos m\varphi & H_r &= -E_{mn} \frac{\sigma + j\omega\varepsilon}{k_c} \frac{m}{k_c r} J_m(k_c r)\sin m\varphi \\ E_\varphi &= E_{mn} \frac{\gamma}{k_c} \frac{m}{k_c r} J_m(k_c r)\sin m\varphi & H_\varphi &= -E_{mn} \frac{\sigma + j\omega\varepsilon}{k_c} J_m'(k_c r)\cos m\varphi \end{aligned} \quad . \tag{3.88}$$

Remarks: 1) If $m >> n$, the fields are concentrated in an annulus near the periphery of the waveguide: these are called whispering gallery modes.

2) We have assumed an azimuthal variation as cos $m\varphi$ for H_z and E_z. When $m \neq 0$, we can as well take sin $m\varphi$ which corresponds to the other polarization of the wave. Since both polarizations have the same cutoff frequency, all modes with $m \neq 0$ are doubly degenerate. This degeneracy is due to the rotational symmetry of the waveguide about the z-axis; it is broken as soon as the rotational symmetry disappears (for example, in an elliptical waveguide).

3.11.3 Coaxial transmission line

The waveguide cross section is the space between two coaxial circular cylinders of radii a, b (with $a < b$). Since this domain is multiply connected with $N = 2$ different conductors, the waveguide can support ($N - 1 = 1$) TEM wave.

The Helmholtz equation is separable in the transverse coordinates r, φ. In the following, $Y_m(z)$ is the Bessel function of the second kind of order m and argument z.

TEM wave

The electrostatic potential V between the cylinders is

$$V = V_o \log\frac{r}{a} \quad \text{with} \quad V_a = 0\ , \quad V_b = V_o \log\frac{b}{a}\ . \tag{3.89}$$

From (3.36) and (3.38), $$E_r = -\frac{V_o}{r} \qquad H_\varphi = \frac{1}{\zeta}E_r = -\frac{V_o}{\zeta r}\ . \tag{3.90}$$

The total current on the outer conductor is

$$I_b = -\oint H_\varphi ds = \frac{2\pi}{\zeta} V_o \quad \text{and} \quad \sum_{i=1}^{2} I_i = 0\ .$$

The characteristic impedance of the line is thus

$$Z_c = \frac{V_b - V_a}{I_b} = \frac{\zeta}{2\pi} \log\frac{b}{a}\ . \tag{3.91}$$

H waves

$$H_z = H_{mn}\left[\frac{J_m(k_c r)}{J_m'(k_c a)} - \frac{Y_m(k_c r)}{Y_m'(k_c a)}\right]\cos m\varphi \quad \text{where} \quad \begin{vmatrix} J_m'(k_c b) & Y_m'(k_c b) \\ J_m'(k_c a) & Y_m'(k_c a) \end{vmatrix} = 0 \tag{3.92}$$

$$\boxed{k_c^2(b-a)^2 \approx (n\pi)^2 + \left(4m^2+3\right)\left(\frac{b-a}{b+a}\right)^2 \qquad m = 0,\ 1,\ 2, \dots \quad n = 1,\ 2, .}$$

For $n = 0$ this relation must be replaced by

$$k_c^2(b+a)^2 \approx 4m^2\left[1 - \frac{1}{3}\left(\frac{b-a}{b+a}\right)^2 + \frac{4}{5.9}\left(6m^2-1\right)\left(\frac{b-a}{b+a}\right)^4 + \frac{4}{3.5.7.9}\left(62m^2-11\right)\left(\frac{b-a}{b+a}\right)^6 + \dots\right]^{-1}$$

For historical reasons these modes are designated as $H_{m,n+1}$ when $m \neq 0$, although some authors designate them with the more logical rotation H_{mn} ([20], p. 329).

The mode with the lowest cut-off frequency, apart from $H_z = C$ (for $m = 0$, $n = 0$, $k_c = 0$), is the H_{11} mode.

E waves

$$E_z = E_{mn}\left[\frac{J_m(k_c r)}{J_m(k_c a)} - \frac{Y_m(k_c r)}{Y_m(k_c a)}\right]\cos m\varphi \quad \text{where} \quad \begin{vmatrix} J_m(k_c b) & Y_m(k_c b) \\ J_m(k_c a) & Y_m(k_c a) \end{vmatrix} = 0 \tag{3.93}$$

$$\boxed{k_c^2(b-a)^2 \approx (n\pi)^2 + \left(4m^2-1\right)\left(\frac{b-a}{b+a}\right)^2 \qquad m = 0,\ 1,\ 2, \dots \quad n = 1,\ 2, .}$$

The mode with the lowest cut-off frequency is the E_{01} mode.

3.11.4 Radial line

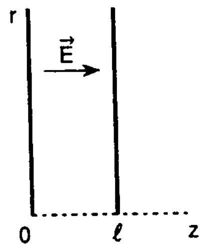


Fig. 3.5 E wave in a radial line

A radial line is the space between two parallel plates a distance ℓ apart, where an electromagnetic wave propagates radially (outwards or inwards). The basic waves are the same as in a coaxial guide, except that in a coaxial guide we have standing waves in the radial direction and travelling waves in the longitudinal direction, whereas in a radial line we have travelling waves in the radial direction and standing waves in the longitudinal direction.

In the radial direction, an inward travelling wave is represented by a Hankel function $H_m^{(1)}(k_r r)$; an outward travelling wave is represented by a Hankel function $H_m^{(2)}(k_r r)$. These functions are singular at $r = 0$. Along z, the propagation constant β must take the values

$$\beta = \frac{n\pi}{\ell} \qquad n = 0,1,2... \tag{3.94}$$

from the boundary conditions at $z = 0$ and $z = \ell$.

Along r, the propagation constant k_r is then given by (3.5) as

$$k_r^2 = k^2 - \beta^2 \tag{3.95}$$

H waves (with respect to the z-axis)

We shall write the fields for a radially outgoing wave; but the formulae are the same for an ingoing wave.

$$H_z = H_{mn} H_m^{(2)}(k_r r) \cos m\varphi \ \sin \beta z \qquad n = 1,2... \tag{3.96}$$

Decomposing $\sin \beta z$ into two travelling waves, from (3.10) we obtain

$$E_r = H_{mn} \frac{j\omega\mu}{k_r} \frac{m}{k_r r} H_m^{(2)}(k_r r) \sin m\varphi \sin\beta z \qquad H_r = H_{mn} \frac{\beta}{k_r} H_m^{(2)'}(k_r r) \cos m\varphi \cos\beta z$$

$$E_\varphi = H_{mn} \frac{j\omega\mu}{k_r} H_m^{(2)'}(k_r r) \cos m\varphi \sin\beta z \qquad H_\varphi = -H_{mn} \frac{\beta}{k_r} \frac{m}{k_r r} H_m^{(2)}(k_r r) \sin m\varphi \cos\beta z \tag{3.97}$$

E waves (with respect to the z-axis)

$$E_z = E_{mn} H_m^{(2)}(k_r r) \cos m\varphi \cos\beta z \qquad n = 0,1,2... \tag{3.98}$$

Decomposing $\cos \beta z$ into two travelling waves, from (3.10) we obtain

$$E_r = -E_{mn} \frac{\beta}{k_r} H_m^{(2)'}(k_r r) \cos m\varphi \ \sin\beta z \qquad H_r = -E_{mn} \frac{\sigma + j\omega\varepsilon}{k_r} \frac{m}{k_r r} H_m^{(2)}(k_r r) \sin m\varphi \ \cos\beta z$$

$$E_\varphi = E_{mn} \frac{\beta}{k_r} \frac{m}{k_r r} H_m^{(2)}(k_r r) \sin m\varphi \sin\beta z \qquad H_\varphi = -E_{mn} \frac{\sigma + j\omega\varepsilon}{k_r} H_m^{(2)'}(k_r r) \cos m\varphi \cos\beta z \tag{3.99}$$

The most important mode is the one with no variation along φ ($m = 0$) and no variation along z ($n = 0$), i.e. the E_{oo} mode. Since $\beta = 0$, from (3.95) $k_r = k$.

This mode has only two field components:

$$E_z = E_{oo} \ H_o^{(2)}(kr) \qquad H_\varphi = -E_{oo} \frac{\sigma + j\omega\varepsilon}{k} H_o^{(2)'}(kr) \tag{3.100}$$

Along r, it is a TEM wave with a wave impedance

$$-\frac{E_z}{H_\varphi} = \frac{k}{\sigma + j\omega\varepsilon} \frac{H_o^{(2)}(kr)}{H_o^{(2)'}(kr)} = -\frac{k}{\sigma + j\omega\varepsilon} \frac{H_o^{(2)}(kr)}{H_1^{(2)}(kr)} \tag{3.101}$$

When $kr \gg 1$,

$$\frac{H_o^{(2)}(kr)}{H_1^{(2)}(kr)} \approx -j$$

and

$$-\frac{E_z}{H_\varphi} \approx \frac{jk}{\sigma + j\omega\varepsilon} = \zeta \quad \text{by (2.64)}\ .$$

Plots of the first modes in waveguides of simple shape can be found in the literature ([24], p. 59 to 84; [25]).

4. IMPEDANCE TRANSFORMATIONS

4.1 Orthogonality of waveguide modes

Because the cross section of a waveguide is two-dimensional, possible modes (and values for k_c^2) are characterized by two indices. In the following we use a single letter n or m to represent the full set of indices.

At a given frequency, the set of all possible (TEM, H and E) modes in a waveguide with a homogeneous isotropic dielectric and perfectly conducting walls form an orthogonal and complete system, which can be used as a basis for expanding the most general fields in the waveguide ([26], p. 121).

By using the identity $\quad U \cdot \Delta V + \text{grad } U \cdot \text{grad } V = \text{div}\,(U \text{ grad } V) \qquad (4.1)$

it is straightforward to deduce from the Helmholtz equations (3.31) combined with the boundary conditions (3.32) that, for E waves:

$$-k_{cm}^2 \int_S E_{zn}E_{zm}dS + \int_S \text{grad}_\perp E_{zn} \cdot \text{grad}_\perp E_{zm} \; dS = \oint E_{zn}\frac{\partial E_{zm}}{\partial n}ds = 0 \qquad (4.2)$$

hence
$$\left(k_{cn}^2 - k_{cm}^2\right)\int_S E_{zn}E_{zm}dS = 0\ .$$

If $k_{cn}^2 \neq k_{cm}^2$, the eigenvalues are said to be non-degenerate, and

$$\int_S E_{zn}E_{zm}dS = 0 \qquad \text{for} \quad n \neq m\ . \qquad (4.3)$$

If $k_{cn}^2 = k_{cm}^2$, by taking appropriate linear combinations of solutions of the Helmholz equation, one can choose a suitable basis of eigenfunctions E_{zn}, E_{zm} such that (4.3) still holds.

Introducing (4.3) in (4.2) yields

$$\int_S \text{grad}_\perp E_{zn} \cdot \text{grad}_\perp E_{zm} \; dS = 0 \qquad \text{for} \qquad n \neq m\ . \qquad (4.4)$$

With (3.10) this entails

$$\int_S \vec{E}_{\perp n} \cdot \vec{E}_{\perp m} dS = 0\ , \qquad \int_S \vec{H}_{\perp n} \cdot \vec{H}_{\perp m} dS = 0\ , \qquad \int_S \left[\vec{E}_n \times \vec{H}_m\right] \cdot d\vec{S} = 0\ , \quad n \neq m\ . \qquad (4.5)$$

For H waves, E_z is replaced by H_z in (4.3) and (4.4), whereas (4.5) remains unchanged.

Since the Helmholz equation and the boundary conditions (3.32) are real, the eigenfunctions E_z, H_z may be taken as real quantities. From (3.10) it then follows that $\vec{E}_{\perp}$, $\vec{H}_{\perp}$, have a constant phase throughout a cross section S of the waveguide; therefore (4.3) and (4.5) may also be written as

$$\int_S E_{zn} E^*_{zm} dS = 0 \qquad \int_S H_{zn} H^*_{zm} dS = 0$$

$$\int_S \vec{E}_{\perp n} \cdot \vec{E}^*_{\perp m} dS = 0 \qquad \int_S \vec{H}_{\perp n} \cdot \vec{H}^*_{\perp m} dS = 0 \tag{4.6}$$

$$\int_S \left[\vec{E}_n \times \vec{H}^*_m\right] \cdot d\vec{S} = 0 \qquad \text{for } n \neq m \qquad \text{(valid with or without *)} .$$

By using the identity $[\text{grad } U \times \text{grad } V] = \text{curl } (U \text{ grad } V)$ which entails

$$\int_S [\text{grad } U \times \text{grad } V] \cdot d\vec{S} = \oint U \frac{\partial V}{\partial s} ds = -\oint V \frac{\partial U}{\partial s} ds \tag{4.7}$$

it is easy to see that the orthogonality relations (4.6) also remain valid when n is an E wave while m is an H wave.

Finally, when one of the waves is TEM (in a multiply connected domain), one uses (3.36) to (3.38) and (3.10) together with the identities (4.1) and (4.7) to prove the validity of (4.6) in this case also. If n and m are both TEM waves, they can be chosen to be orthogonal since they correspond to the same degenerate eigenvalue $k_c^2 = 0$.

4.2 Reflection at an obstacle

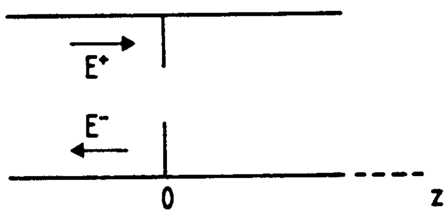


Fig. 4.1 An obstacle in the waveguide at $z = 0$

Wave travelling in the positive z-direction

$$\begin{cases} \vec{E}^+ = \sum\limits_n A_n^+ \left[\vec{E}_{\perp n} + E_{zn} \vec{1}_z\right] e^{-\gamma_n z} \\ \vec{H}^+ = \sum\limits_n A_n^+ \left[\vec{H}_{\perp n} + H_{zn} \vec{1}_z\right] e^{-\gamma_n z} \end{cases} \tag{4.8}$$

$$\vec{E}_{\perp n} = Z_{on} \left[\vec{H}_{\perp n} \times \vec{1}_z\right] \tag{4.9}$$

where Z_o is given by (3.20) for an H wave and by (3.22) for an E wave; Z_o is simply ζ for a TEM wave.

Wave travelling in the negative z-direction

By changing the sign of γ in (3.10) or by considering the image of $\vec{E}^+$ in a symmetry plane, we may write

$$\begin{cases} \vec{E}^- = \sum\limits_n A_n^- \left[\vec{E}_{\perp n} - E_{zn} \vec{1}_z\right] e^{\gamma_n z} \\ \vec{H}^- = \sum\limits_n A_n^- \left[-\vec{H}_{\perp n} + H_{zn} \vec{1}_z\right] e^{\gamma_n z} . \end{cases} \tag{4.10}$$

Consider a wave E^+ impinging on an obstacle at $z = 0$. In general, the boundary conditions on the obstacle require the presence of all possible modes in its vicinity. However, if the frequency is low enough (lower than the second cut-off frequency), only the mode having the lowest cut-off frequency (dominant mode) will propagate away from the obstacle, so that, at some distance, the reflected wave E^- will essentially reduce to the dominant mode.

A particular mode is completely determined by the amplitudes of $\vec{E}_\perp$ or $\vec{H}_\perp$ in the two travelling waves at some point in the cross section of the waveguide. Having chosen such a point, one may write

$$E_\perp^+ = E_o^+ e^{-\gamma z} \qquad H_\perp^+ = \frac{E_\perp^+}{Z_o}$$
$$E_\perp^- = E_o^- e^{\gamma z} \qquad H_\perp^- = -\frac{E_\perp^-}{Z_o} . \tag{4.11}$$

The wave impedance Z_o is defined with respect to a direction of propagation, and changes sign with the latter. The total wave in this particular mode reads

$$\begin{cases} E_\perp = E_o^+ e^{-\gamma z} + E_o^- e^{\gamma z} \\ H_\perp = \dfrac{1}{Z_o}\left[E_o^+ e^{-\gamma z} - E_o^- e^{\gamma z}\right] \end{cases} . \tag{4.12}$$

Analogy with transmission lines: $V \quad I \quad Z_c$ or Z_o

Transverse fields in waveguides: $E_\perp \quad H_\perp \quad Z_o$

Reflection coefficient (of the transverse electric field)

Since in the following we only use the transverse fields, we drop the subscript $\perp$. By definition, the reflection coefficient of the transverse electric field is

$$\rho = \frac{E^-}{E^+} . \tag{4.13}$$

In particular, $$\rho_o = \frac{E_o^-}{E_o^+} = |\rho_o|\, e^{j\phi_o} .$$

For the transverse magnetic field, the reflection coefficient is $(-\rho)$.

From (4.11), $$\rho = \rho_o \; e^{2\gamma z} = |\rho_o|\, e^{2\alpha z + j(2\beta z + \phi_o)} . \tag{4.14}$$

Let us remember that since the reflecting obstacle is located at $z = 0$ (see Fig. 4.1), in (4.14) z is always negative. Therefore, when $\alpha > 0$, $|\rho|$ decreases toward the RF generator; when $\alpha = 0$, $|\rho|$ is independent of z.

4.3 Standing waves

At every position z in the waveguide,

$$E = E^+ (1 + \rho) \qquad H = H^+(1 - \rho) . \tag{4.15}$$

If the attenuation is zero, $|E^+|$, $|H^+|$ and $|\rho|$ do not depend on z. Then, along the waveguide:

$$|E|_{max} = \left|E^+\right| \left(1 + |\rho|\right) \qquad |E|_{min} = \left|E^+\right| \left(1 - |\rho|\right) . \tag{4.16}$$

Voltage standing wave ratio (VSWR)

The VSWR is defined as $$S = \frac{|E|_{max}}{|E|_{min}} \geq 1 \text{ along the waveguide .} \tag{4.17}$$

If the attenuation is negligible, (4.16) yields

$$S = \frac{1+|\rho|}{1-|\rho|} \quad \text{or} \quad |\rho| = \frac{S-1}{S+1} \, . \tag{4.18}$$

The voltage standing wave ratio S is thus a measure of $|\rho|$.

When $|\rho| = 1$, $S = \infty$: this is the case of a pure standing wave with total reflection, where the reflected wave has the same amplitude as the incident wave.

When $\rho = 0$, $S = 1$: this is the case of a pure travelling wave, and the waveguide is said to be matched. From (4.14), this condition is independent of z.

Amplitude variation of E, H along the waveguide

For simplicity's sake we assume that the attenuation α is negligible. From (4.14) and (4.15):

$$\frac{|E|^2}{|E^+|^2} = 1+|\rho|^2 + 2|\rho|\cos(2\beta z + \phi_o) = (1-|\rho|)^2 + 4|\rho|\cos^2\left(\beta z + \frac{\phi_0}{2}\right)$$

$$\frac{|H|^2}{|H^+|^2} = 1+|\rho|^2 - 2|\rho|\cos(2\beta z + \phi_o) = (1-|\rho|)^2 + 4|\rho|\sin^2\left(\beta z + \frac{\phi_0}{2}\right)$$

$|E|$ is maximum when $|H|$ is minimum, and vice versa (see Fig. 4.2).

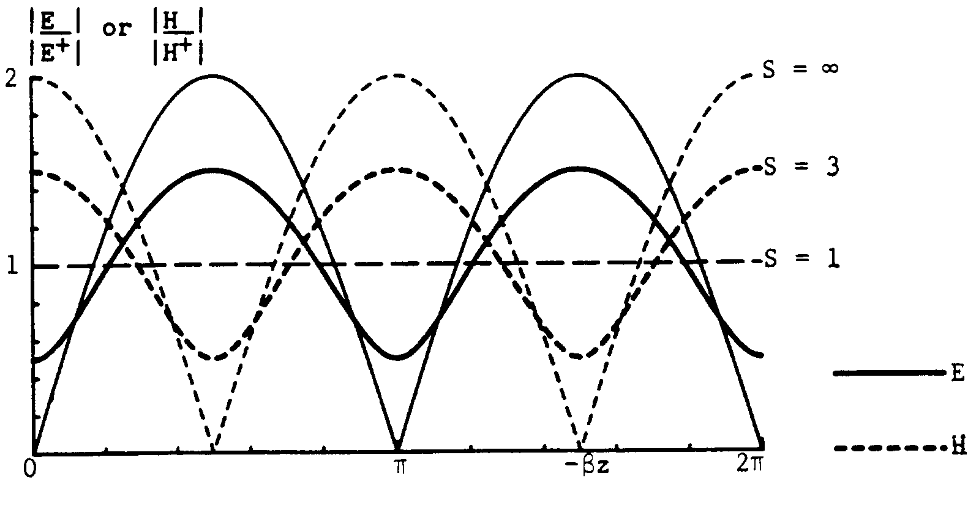


Fig. 4.2 Amplitude variation of E and H with βz. In the figure, for $S = \infty$ we have taken $\phi_o = \pi$ (i.e. $\rho_o = -1$)

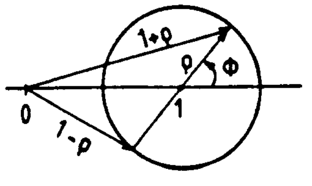


Fig. 4.3 Variation of $(1 \pm \rho)$ in the complex plane

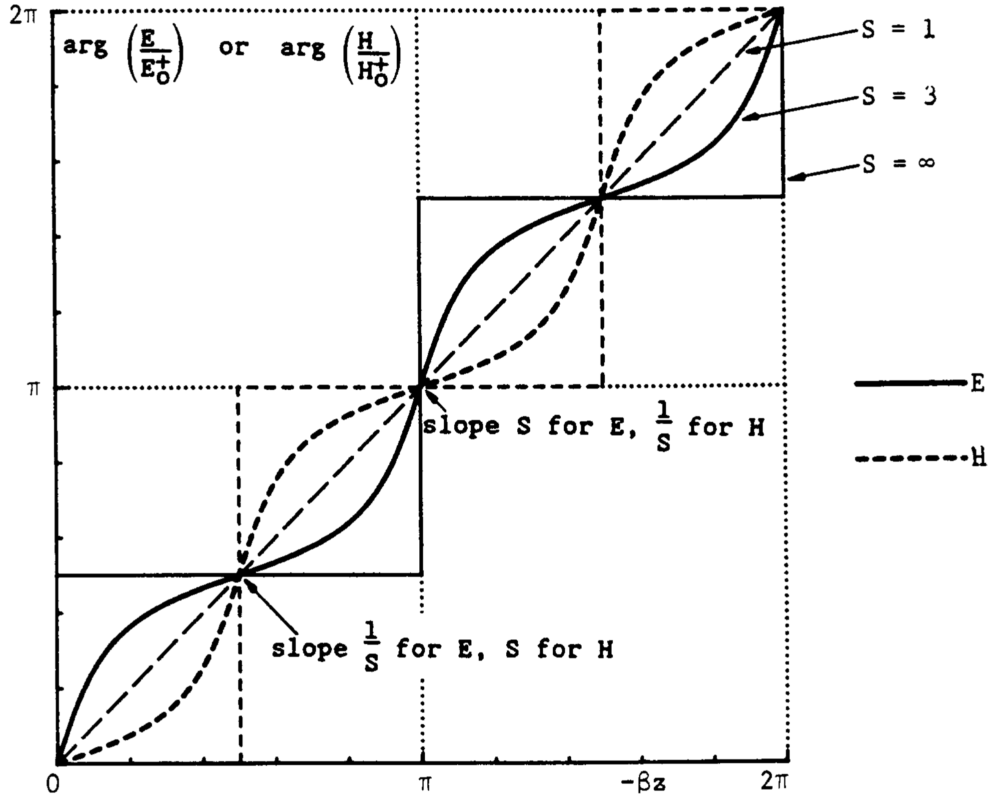


Fig. 4.4 Phase variation of E and H with βz. The figure is drawn for the case $\phi_o = \pi$.

Phase variation of E, H along the waveguide

From (4.14) when $\alpha = 0$, $\rho = |\rho_o|\, e^{j\phi}$ where $\phi = 2\beta z + \phi_o$

The variation of $(1 \pm \rho)$ with ϕ is shown in Fig. 4.3. The phase variation of E and H with βz follows from (4.15); it is shown in Fig. 4.4. When $S = \infty$, the phase of E (or H) is constant except for jumps of 180° at each node of the standing wave. Moreover, E and H are always in quadrature.

4.4 Impedance transformations

At a distance z along the waveguide, the impedance Z and the admittance Y are defined in terms of the transverse fields by:

$$Z = \frac{E}{H} \qquad Y = \frac{1}{Z} = \frac{H}{E} \tag{4.19}$$

Using (4.15) and (4.11) this becomes

$$Z = \frac{E^+}{H^+}\frac{1+\rho}{1-\rho} = Z_o \frac{1+\rho}{1-\rho} \ . \tag{4.20}$$

Z, like Z_o, is defined with respect to a direction of propagation; it changes sign with the latter, as $[\vec{E} \times \vec{H}^*]_z$. The direction of propagation is chosen to be positive from the generator towards the load. As is evident from (4.20), the only quantity which appears in the equations is the normalized impedance $z = Z/Z_o$ or the normalized admittance $y = Y/Y_o = 1/z$.

Relation between z, y, ρ

From (4.20), at every position z along the waveguide,

$$z = \frac{1+\rho}{1-\rho} \qquad \rho = \frac{z-1}{z+1}$$
$$y = \frac{1-\rho}{1+\rho} \qquad \rho = \frac{1-y}{1+y} \tag{4.21}$$

These bilinear transformations in the complex plane transform circles into circles (a straight line being considered as a circle of infinite radius).

Smith Chart

The Smith chart is the disk $|\rho| \leq 1$ in the complex ρ-plane, where the circles of constant $r = \mathrm{Re}(z)$ or $g = \mathrm{Re}(y)$, and $x = \mathrm{Im}(z)$ or $b = \mathrm{Im}(y)$ have been drawn and labelled (see Fig. 4.5). From (4.14) with $\alpha = 0$, when z is varied, $\rho = \rho_o\, e^{2j\beta z}$ simply rotates around the origin in the ρ-plane; so one immediately reads the transformation of z or y from A to B.

Normally the ρ-plane is presented in a way which corresponds to z, with the points $\rho = -1, 0, +1$ appearing from left to right on the real axis. It is seen from (4.21) that adding 180° to the phase of ρ transforms z into y; therefore the same chart can be used for y, but in that case the points $\rho = -1, 0, +1$ appear from right to left on the real axis.

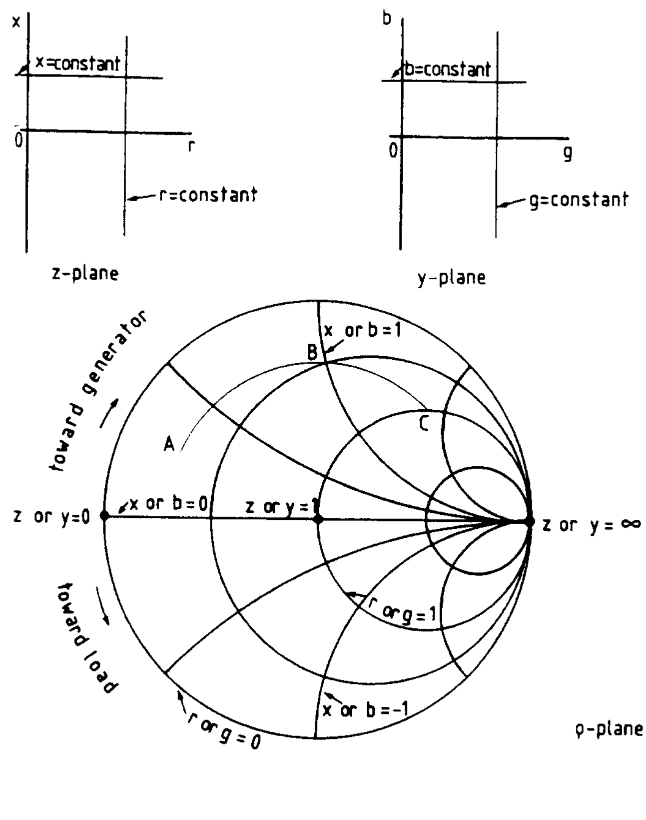


Fig. 4.5 Smith Chart: Transforming from z- or y-plane to ρ-plane

Chain matrix of a length ℓ of waveguide

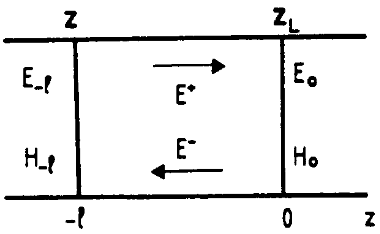


Fig. 4.6 A waveguide section of length ℓ

The chain matrix is the matrix A which relates the fields $E_{-\ell}$, $H_{-\ell}$ at the input of the waveguide section to the fields E_o, H_o at the output.

From (4.12) we have

$$\begin{cases} E_{-\ell} = E_o^+ e^{\gamma\ell} + E_o^- e^{-\gamma\ell} \\ H_{-\ell} = \dfrac{1}{Z_o}\left[E_o^+ e^{\gamma\ell} - E_o^- e^{-\gamma\ell}\right] \end{cases} \qquad \begin{cases} E_o = E_o^+ + E_o^- \\ H_o = \dfrac{1}{Z_o}\left[E_o^+ - E_o^-\right] . \end{cases}$$

Therefore $\qquad E_o^+ = \frac{1}{2}(E_o + Z_o H_o) \qquad\qquad E_o^- = \frac{1}{2}(E_o - Z_o H_o)$

and

$$\begin{bmatrix} E_{-\ell} \\ H_{-\ell} \end{bmatrix} = \begin{bmatrix} \cosh(\gamma\ell) & Z_o \sinh(\gamma\ell) \\ \dfrac{1}{Z_o}\sinh(\gamma\ell) & \cosh(\gamma\ell) \end{bmatrix} \begin{bmatrix} E_o \\ H_o \end{bmatrix} . \tag{4.22}$$

The chain matrix thus reads

$$A(\ell) = \begin{bmatrix} \cosh(\gamma\ell) & Z_o \sinh(\gamma\ell) \\ \dfrac{1}{Z_o}\sinh(\gamma\ell) & \cosh(\gamma\ell) \end{bmatrix} . \tag{4.23}$$

It is readily verified that

$$A^{-1}(\ell) = A(-\ell) \tag{4.24}$$

and

$$\det\ A = \det\ A^{-1} = 1 . \tag{4.25}$$

The last equation is simply an expression of reciprocity, because the relation between the chain matrix A and the scattering matrix S is such that

$$\det\ A = \frac{S_{12}}{S_{21}} .$$

Impedance transformation by a length ℓ of waveguide

From (4.19), at $z = -\ell$

$$z = \frac{E_{-\ell}}{Z_o H_{-\ell}} \quad \text{whereas at } z = 0, \quad z_L = \frac{E_o}{Z_o H_o}$$

where in z_L the subscript L stands for "load".

From (4.22):

$$z = \frac{z_L + \tanh(\gamma \ell)}{1 + z_L \tanh(\gamma \ell)} \qquad y = \frac{y_L + \tanh(\gamma \ell)}{1 + y_L \tanh(\gamma \ell)} \tag{4.26}$$

Particular cases:

1) $z_L = 1$ then $z = 1$ for any ℓ. This corresponds to $\rho = 0$, i.e. to a matched waveguide.

2) $z_L = 0$ then $z = \tanh(\gamma \ell)$. If $\alpha = 0, \gamma = j\beta$ and $Z = Z_o \, j \tan(\beta \ell)$ (4.27)

Such a short-circuited length of waveguide can provide any reactance according to the value of ℓ.

Impedance matching

When a waveguide is matched:

1. Its input impedance is independent of its length.
2. There is no reflected power towards the generator.
3. VSWR = 1, which means that there is no voltage nor current peak along the waveguide.

There are several means for matching a given load Y_L to a waveguide (at a given frequency). For example:

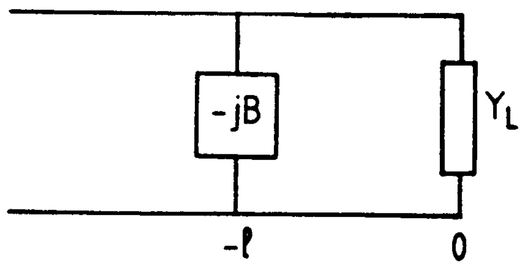


Fig. 4.7 Matching of a load Y_L with a shunt susceptance at $z = -\ell$

By using a properly chosen length ℓ of guide to arrive at C (Fig. 4.5), one crosses the $g = 1$ circle in the Smith chart, where the admittance appears as $Y_o + jB$. It is then sufficient to connect at that point a shunt susceptance $-jB$, which can be obtained with a short-circuited length of waveguide (called "stub").

In practice, it is easy to vary the stub length by using a sliding short-circuit in a piece of waveguide, but it is not so easy to vary the distance between the stub and the load. If this distance is fixed, several stubs are generally needed to match the load; in fact it is always possible to match any load by using three equidistant stubs at a fixed distance of 3/8 λ_g apart.

5. PERIODICALLY LOADED WAVEGUIDES

5.1 Chain matrix

The simplest example of a periodic structure is a waveguide which is periodically loaded with lossless obstacles (infinitely thin diaphragms, ...). For the dominant mode, these infinitely thin obstacles behave as series or shunt reactances. We assume that the cell length L is long enough for the evanescent modes from one obstacle to be negligible at the position of the next obstacle. In the following we consider the case where the obstacle is equivalent to a shunt susceptance Y.

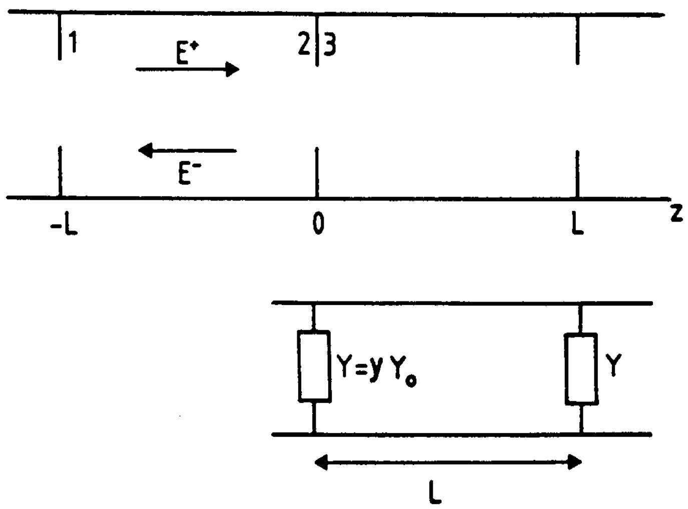


Fig. 5.1 A waveguide with periodic loading of spacing L

Boundary conditions between 2 and 3 (see Fig. 5.1):

$$\begin{cases} E_2 = E_3 & \text{(continuity of transverse electric field)} \\ \dfrac{H_2}{E_2} = \dfrac{H_3}{E_3} + Y & \text{(definition of Y)}. \end{cases} \tag{5.1}$$

Then

$$\begin{bmatrix} E_3 \\ H_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -yY_o & 1 \end{bmatrix} \begin{bmatrix} E_2 \\ H_2 \end{bmatrix}$$

If γ_o is the propagation constant in the *unloaded* waveguide, from (4.22) and (4.24) we have:

$$\begin{bmatrix} E_2 \\ H_2 \end{bmatrix} = \begin{bmatrix} \cosh(\gamma_o L) & -\dfrac{1}{Y_o}\sinh(\gamma_o L) \\ -Y_o \sinh(\gamma_o L) & \cosh(\gamma_o L) \end{bmatrix} \begin{bmatrix} E_1 \\ H_1 \end{bmatrix}$$

The chain matrix T for a full period of the structure is such that

$$\begin{bmatrix} E_3 \\ H_3 \end{bmatrix} = T \begin{bmatrix} E_1 \\ H_1 \end{bmatrix} \tag{5.2}$$

hence

$$T = \begin{bmatrix} 1 & 0 \\ -yY_o & 1 \end{bmatrix} \begin{bmatrix} \cosh(\gamma_o L) & -\dfrac{1}{Y_o}\sinh(\gamma_o L) \\ -Y_o \sinh(\gamma_o L) & \cosh(\gamma_o L) \end{bmatrix} \tag{5.3}$$

and from (4.25), $\det T = 1$ (5.4)

By Floquet's theorem, there are particular solutions of Maxwell's equations, called travelling waves, such that for a translation of one period L along the structure:

$$\begin{bmatrix} E(z+L) \\ H(z+L) \end{bmatrix} = e^{-\gamma L} \begin{bmatrix} E(z) \\ H(z) \end{bmatrix} \tag{5.5}$$

$\gamma = \alpha + j\beta$ is the propagation constant for a wave travelling in the positive z-direction along the periodic structure. (Let us remember that γ_o is the propagation constant for a wave travelling in the positive z-direction along the ***unloaded*** waveguide.) When γL is purely imaginary we put $\gamma L = j\theta$ where θ is the phase-shift per period of the structure.

Comparing (5.5) with (5.2) shows that $e^{-\gamma L}$ is an eigenvalue of the chain matrix T. By (5.4) the product of the two eigenvalues of T is 1; therefore the other eigenvalue is $e^{\gamma L}$, which corresponds to a wave travelling in the negative z-direction along the periodic structure (this is reciprocity). The sum of the two eigenvalues of T is given by

$$e^{\gamma L} + e^{-\gamma L} = \mathrm{Tr}(T)\ . \tag{5.6}$$

5.2 Dispersion diagram

Using (5.3) and (5.6) we obtain the dispersion relation

$$\cosh(\gamma L) = \cos\theta = \cosh(\gamma_o L) + \frac{y}{2}\sinh(\gamma_o L)\ . \tag{5.7}$$

Dependence of Y on frequency

The admittance of a lossless one-port network can be written as an infinite series of terms, each representing the admittance of a series-resonant circuit at a resonant frequency of the network ([27], p. 87):

$$Y = \sum_{n=0}^{\infty} \frac{j\omega}{L_n\left(\omega_n^2 - \omega^2\right)} \tag{5.8}$$

where ω_n is the n^{th} series-resonant frequency of the network, and $L_n > 0$ is the equivalent self-inductance of the network at frequency ω_n. The frequencies ω_n are supposed to be numbered in increasing order, starting with $\omega_o = 0$ (if $L_o < \infty$).

We can rewrite (5.8) as

$$Y = \frac{1}{j\omega L_o} + j\omega\sum_{n=1}^{\infty}\frac{1}{L_n\omega_n^2} + j\omega^3\sum_{n=1}^{\infty}\frac{1}{L_n\left(\omega_n^2 - \omega^2\right)\omega_n^2}.$$

For low frequencies ($\omega << \omega_1$), this expression approximates to

$$Y = \frac{1}{j\omega L_o} + j\omega C_o \quad \text{where} \quad C_o = \sum_{n=1}^{\infty}\frac{1}{L_n\omega_n^2} > 0 \tag{5.9}$$

Dependence of y on γ_o

The susceptance $Y = yY_o$ is independent of the direction of propagation; therefore it is an even function of γ_o. Since Y_o is an odd function of γ_o, y must be an odd function of γ_o.

In the unloaded, lossless guide:

a) for an H wave, from (3.20) we have $Y_o = \gamma_o/(j\omega\mu)$; therefore, using (5.9) and (3.5) we obtain

$$y\gamma_o = \frac{\mu}{L_o} - k^2\frac{C_o}{\varepsilon} = \left(\frac{\mu}{L_o} - k_c^2\frac{C_o}{\varepsilon}\right) + \gamma_o^2\frac{C_o}{\varepsilon}\ . \tag{5.10}$$

Corresponding to $L_o = \infty$, a purely capacitive term $-k^2C_o/\varepsilon$ may be obtained with a dielectric rod across a rectangular waveguide ([28], p. 122). For metallic inductive obstacles, the term μ/L_o is the most important one (at low frequencies); for metallic capacitive obstacles, in all examples published in the literature [24, 28], the bracket in (5.10) vanishes and $y = \gamma_o C_o/\varepsilon$.

b) for an E wave, from (3.22) when $\sigma = 0$, we have $Y_o = j\omega\varepsilon/\gamma_o$; therefore using (5.9) we obtain

$$\frac{y}{\gamma_o} = -\frac{\mu}{k^2 L_o} + \frac{C_o}{\varepsilon} . \qquad (5.11)$$

For a purely capacitive obstacle ($L_o = \infty$), again $y = \gamma_o C_o/\varepsilon$.

We shall here consider only two very simple cases:

1) inductive obstacle (H wave):

$$y = \frac{1}{\gamma_o}\frac{\mu}{L_o} = \frac{1}{\gamma_o L}\cdot 2K_1 . \qquad (5.12)$$

2) capacitive obstacle (H or E wave):

$$y = \gamma_o \frac{C_o}{\varepsilon} = \gamma_o L \cdot 2K_2 . \qquad (5.13)$$

In (5.12) and (5.13), K_1 and K_2 are real, positive, dimensionless constants.

Inductive obstacles

With (5.12) the dispersion relation (5.7) reads

$$\cosh(\gamma L) = \cosh(\gamma_o L) + K_1 \frac{\sinh(\gamma_o L)}{\gamma_o L} \qquad \text{where} \qquad \gamma L = \alpha L + j\beta L . \qquad (5.14)$$

For the lossless unloaded waveguide, $\gamma_o L$ is either real or purely imaginary; therefore $\cosh(\gamma L)$ is always real, which entails

$$\sinh(\alpha L)\sin(\beta L) = 0 \qquad \text{hence} \qquad \alpha L = 0 \quad \text{or} \quad \beta L = n\pi \qquad (5.15)$$

When $\alpha L = 0$, $\gamma L = j\theta$, which corresponds to a passband of the loaded waveguide; whereas $\beta L = n\pi$, $\gamma L = \alpha L + jn\pi$ corresponds to a stopband of the loaded waveguide.

Above the smooth guide cut-off, (5.14) reads

$$\cosh(\gamma L) = \cos(\beta L) = \cos(\beta_o L) + K_1 \frac{\sin(\beta_o L)}{\beta_o L} \qquad \text{with} \quad K_1 > 0 . \qquad (5.16)$$

This relation is plotted in Fig. 5.2 for $K_1 = 4$.

In the unloaded waveguide, when ω increases from 0 in the dispersion diagram, one first follows the ω-axis up to ω_c, afterwards one follows the hyperbola $k^2 = k_c^2 + \beta_o^2$.

For the loaded waveguide, the cut-off frequency of the first passband is reached when

$$\beta_o L = 0 \qquad \text{i.e.} \qquad \frac{\beta_o L}{2}\tan\frac{\beta_o L}{2} = \frac{K_1}{2} .$$

This happens for $0 < \beta_o L < \pi$: it is the beginning of the first passband; when $\beta_o L = \pi$, $\beta L = \pi$ is the end of the first passband. For a slightly higher frequency, $\cosh(\gamma L) < -1$ and $\gamma L = \alpha L + j\pi$: this is the second stopband. For increasing frequency, γL comes back to $j\pi$ when

$$\frac{\beta_o L}{2}\operatorname{ctn}\frac{\beta_o L}{2} = -\frac{K_1}{2} .$$

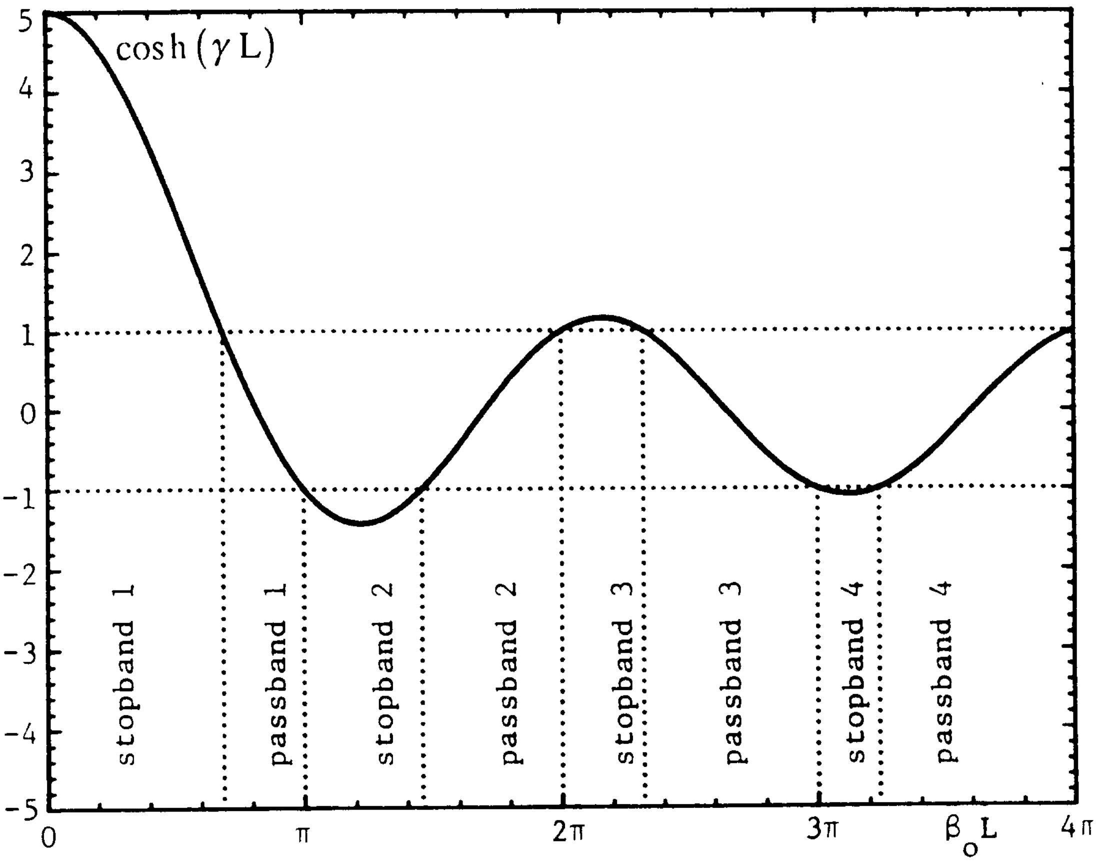


Fig. 5.2 Variation of $\cosh(\gamma L)$ with $\beta_o L$ for inductive obstacles (H wave in unloaded guide). In the figure $K_1 = 4$.

This happens for $\pi < \beta_o L < 2\pi$: it is the beginning of the second passband. When $\beta_o L = 2\pi$, $\beta L = 2\pi$: this is the end of the second passband.

More generally, when $\beta_o L = n\pi$, $\beta L = n\pi$ is the end of the n^{th} passband ($n = 1, 2, ...$).

Capacitive obstacles

With (5.13) the dispersion relation (5.7) reads

$$\cosh(\gamma L) = \cosh(\gamma_o L) + K_2(\gamma_o L)\ \sinh(\gamma_o L) \qquad \text{where} \qquad \gamma L = \alpha L + j\beta L \tag{5.17}$$

Again, $\cosh(\gamma L)$ is always real. Above the smooth guide cut-off, this relation becomes

$$\cosh(\gamma L) = \cos(\beta L) = \cos(\beta_o L) - K_2(\beta_o L)\ \sin(\beta_o L) \qquad \text{with} \quad K_2 > 0 \tag{5.18}$$

This relation is plotted in Fig. 5.3 for $K_2 = 0.2$.

For the loaded waveguide, the first passband starts when $\beta L = 0$, i.e. when $\beta_o L = 0$; the cut-off frequency of the smooth guide is thus also the cut-off frequency of the loaded guide. The first passband ends when $\beta L = \pi$, i.e. when

$$\frac{\beta_o L}{2} \tan \frac{\beta_o L}{2} = \frac{1}{2K_2} \ ;$$

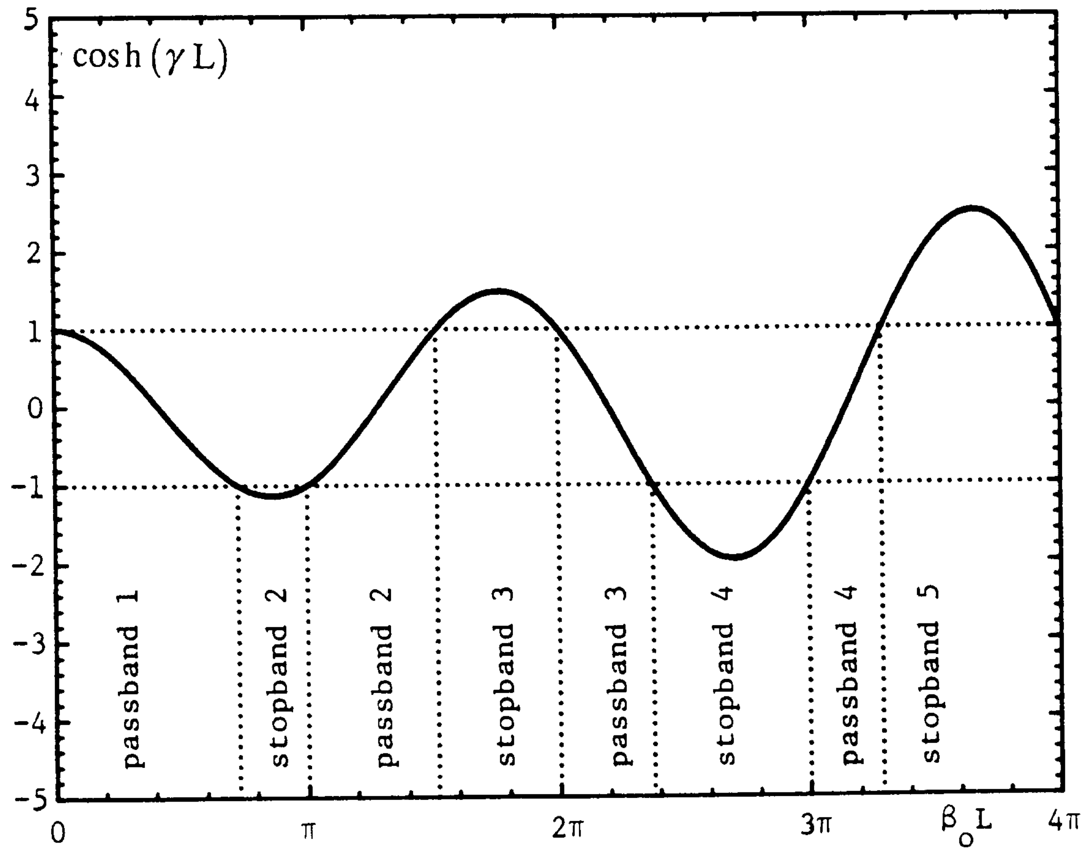


Fig. 5.3 Variation of cosh(γL) with $\beta_o L$ for capacitive obstacles (H or E wave in unloaded guide). In the figure $K_2 = 0.2$.

this happens for $0 < \beta_o L < \pi$. For a slightly higher frequency, $\gamma L = \alpha L + j\pi$: this is the second stopband. γL comes back to $j\pi$ when $\beta_o L = \pi$, which is the beginning of the second passband. The second passband ends when $\beta L = 2\pi$, i.e. when

$$\frac{\beta_o L}{2} \operatorname{ctn} \frac{\beta_o L}{2} = -\frac{1}{2K_2} \; ;$$

this happens for $\pi < \beta_o L < 2\pi$. Generally, when $\beta_o L = n\pi$, $\beta L = n\pi$ is the beginning of the $(n+1)^{\text{th}}$ passband (n = 0, 1, 2, ...).

Finally, the (k,β) dispersion diagram is obtained by using β_o as a parameter and computing k from $k^2 = k_c^2 + \beta_o^2$ while β is deduced from (5.16) or (5.18); the result is shown in Fig. 5.4.

Remarks

1) From Fig. 4.2, the distance z between nodes in a standing wave with infinite VSWR is such that $\beta_o z = \pi$. Therefore, when $\beta_o L = n\pi$, for standing waves with $S = \infty$ in the smooth guide, L is an exact multiple of the distance between nodes. One may then insert infinitely thin diaphragms at the nodes of $E_\perp$ without perturbing the fields, which yields $\beta L = n\pi$ for the loaded guide at the same frequency. This explains why in the dispersion diagram, the points $\beta_o L = n\pi$ of the unloaded guide also belong to the dispersion curves of the loaded guide.

There is however one exception to this rule: the $\beta_o L = 0$ point of an H wave in the smooth guide (see Fig. 5.2). Indeed, such a wave would have $H_z \neq 0$ independent of z (since $\beta = 0$), which would violate the boundary conditions on the diaphragms.

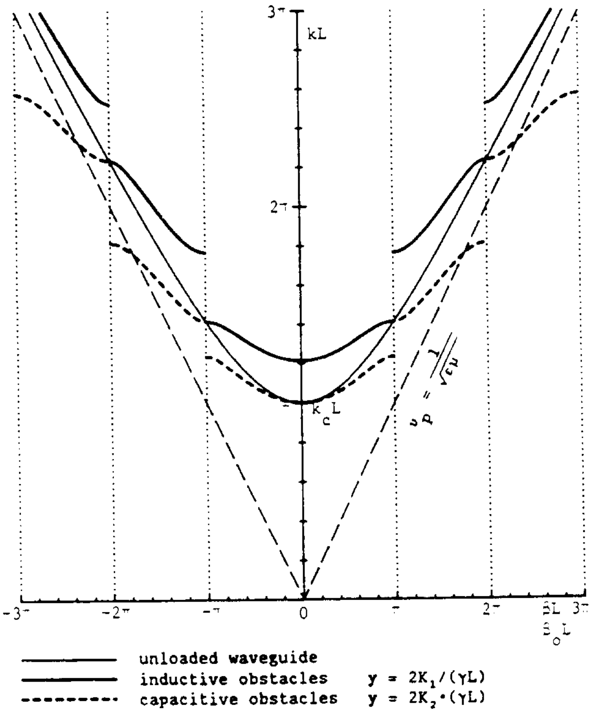


Fig. 5.4 Dispersion diagram for a periodically loaded waveguide. $\beta_o L$ and βL are the phase shifts per cell in the unloaded and a loaded waveguide respectively. For the sake of clarity, the reflection of the curves about the vertical lines $\beta L = n\pi$ has not been drawn. The figure corresponds to the case $k_c L = \pi$, $K_1 = 4$, $K_2 = 0.2$.

2) The dispersion relation (5.7) only determines γL up to a change of sign, and βL modulo 2π. It follows that the dispersion diagram is symmetrical with respect to the vertical axis $\beta L = 0$, and is periodic with period 2π in βL; it is therefore also symmetrical with respect to all vertical lines $\beta L = n\pi$, where n is any integer. As a consequence, it is sufficient to consider the interval $0 < \beta L < \pi$.

Because of the 2π periodicity of the dispersion curves for the loaded waveguide, any straight line $\upsilon_p = \omega/\beta = \upsilon$ crosses every passband at one point. Therefore, whatever a particle velocity υ may be, in every passband of a loaded guide there is a frequency for which the phase velocity is equal to υ. This is in contrast with the unloaded guide where $\upsilon_p > 1/\sqrt{\varepsilon\mu}$.

3) When the loading y of the waveguide is increased, for example by decreasing the hole radius of the irises (which are capacitive obstacles for an E wave), the width of the stopbands increases while the width of the passbands decreases. For maximum loading (i.e. no hole in the irises), the passbands have zero width; the dispersion curves reduce to horizontal lines passing through the points $\beta_o L = n\pi$ of the unloaded guide, which is natural because the cells in the loaded guide are then completely decoupled.

Resonant coupling and backward waves

Although the above examples are over simplified, they yield a correct qualitative picture of the dispersion diagram for a periodic structure. They become increasingly inaccurate at higher frequencies, when several modes can propagate in the smooth waveguide, and when the approximation (5.9) has to be replaced by the correct expression (5.8) for the shunt susceptance of the obstacles. In particular, (5.8) must be used when the first resonance of the obstacles is close to the cut-off frequency of the smooth guide; the obstacles then produce a resonant coupling between cells, thereby modifying qualitatively the dispersion diagram. An example is a disk-loaded waveguide with slots in the irises, i.e. the slotted iris structure, for which the first passband is of backward wave type ([29], p. 681). The same is true for the travelling wave accelerating structure of the CERN SPS [30], which is a circular waveguide periodically loaded with horizontal bars; the bars behave as $\lambda/2$ resonators whose resonant frequency lies below the E_{01} mode cut-off of the cylindrical envelope.

Coupled modes

Most often, in the lower passbands, the frequency is increasing or decreasing monotonously within the interval $0 < \beta L < \pi$. However, even in lossless periodic structures, dispersion curves may exhibit a local maximum or minimum within the passband, i.e. when $\beta L \neq n\pi$. In the stopband which starts at such local extremum, $\alpha L \neq 0$ while $\beta L \neq n\pi$; this means that $\cosh(\gamma L)$ is then complex [31].

Such a case occurs for some range of the hole diameter in a disk-loaded waveguide, for the hybrid EH_{11} wave which is used in RF separators ([19], p. 252). It occurs more often at higher frequencies, when several modes propagate in the unloaded waveguide and their dispersion curves cross when folded into the interval $0 < \theta < \pi$. In Fig. 5.5 are shown the dispersion curves for two modes (E_{01}, E_{02}) in an unloaded circular waveguide. These curves are the dispersion curves for vanishingly small loading of the guide. When the loading is increased, the curves are progressively deformed, as it appears in Fig. 5.4.

It can be shown ([32], Chap. 6,7, 8)(*) that if they correspond to modes which are coupled through the lossless obstacles, the curves do not cross (as the unperturbed curves do at points A, B, C in Fig. 5.5), but they split into two curves, like the two branches of a hyperbola. If the dispersion curves of the unloaded waveguide cross with group velocities of opposite signs, one branch of the dispersion curve of the loaded waveguide has a local maximum and the other has a local minimum. In the stopband between these local extrema, $\cosh(\gamma L)$ takes two different complex conjugate values corresponding to the two coupled modes (see Fig. 5.6). The crossing point is locally a centre of symmetry for the dispersion curves. Loading the waveguide not only provides coupling between modes, but also shifts the frequency of the crossing point.

If k_1 and k_2 (with $k_1 < k_2$) are the frequencies at the extrema, the dispersion curves are locally represented by

$$\gamma = j\left[a\left(k - \frac{k_1 + k_2}{2}\right) + b \pm c\sqrt{(k-k_1)(k-k_2)}\right] \tag{5.19}$$

where a, b, c are real coefficients.

In the passbands $k < k_1$ or $k > k_2$, we have

$$\alpha = 0\ , \qquad \beta = a\left(k - \frac{k_1 + k_2}{2}\right) + b \pm c\sqrt{(k-k_1)(k-k_2)} \tag{5.20}$$

In the stopband $k_1 < k < k_2$ we have

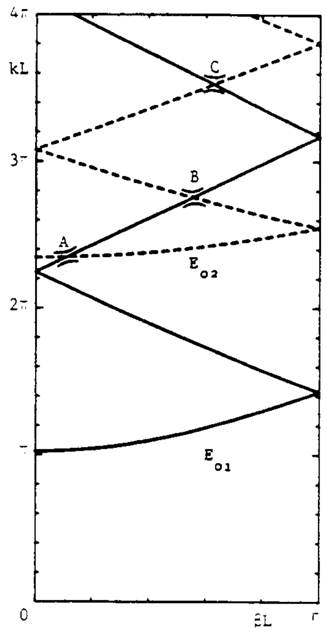
$$\alpha = \mp c\sqrt{(k-k_1)(k_2-k)}\ , \qquad \beta = a\left(k - \frac{k_1 + k_2}{2}\right) + b \tag{5.21}$$

β of (5.21) is represented by the dotted line in Fig. 5.6(b).

If $\gamma = \alpha + j\beta$ is a complex propagation constant at a given frequency, so are $-\gamma$, γ^* and hence $-\gamma^*$. Indeed, $-\gamma$ represents a wave travelling in the opposite direction, which (except for the sign) has the same propagation constant when the structure does not contain nonreciprocal media. When losses are neglected, the Helmholtz equation and the boundary conditions for the fields are real, so that the complex conjugate of a solution is also a solution: therefore γ^* is also a possible propagation constant.

Summarizing, if $\gamma = \alpha + j\beta$ is a propagation constant, so are $\pm\alpha + j\beta$ and $\pm\alpha - j\beta$. In the half-plane $\beta > 0$ they correspond to the two waves (5.21) when there are extrema at k_1, k_2 inside the interval $0 < \beta L < \pi$. In the opposite case, there is only one wave for each direction of propagation; this requires that either α could not be distinguished from $-\alpha$ (which means that $\alpha = 0$), or that βL could not be distinguished from $-\beta L$ modulo 2π (which means that $\beta L = n\pi$): in the first situation we are in a normal passband (without extremum), in the second situation we are in a normal stopband at $\beta L = n\pi$.

(*) It should be noted that k in this reference is our β.



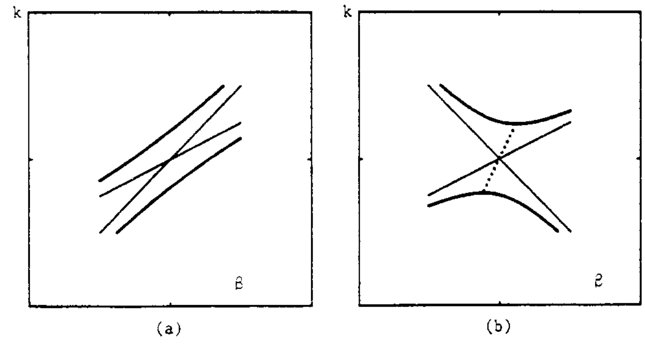


Fig. 5.5 Dispersion curves for an unloaded waveguide, folded into the interval $0 < \beta L < \pi$. The figure corresponds to a circular waveguide of radius b, with $L/b = 4/3$.

Fig. 5.6 Close-up view of crossing points. The dispersion curves of the unloaded waveguide cross with group velocities of the same sign in (a), of opposite signs in (b). On the dotted line in (b), $\cosh(\gamma L)$ is complex.

If there is no stopband inside the interval $0 < \beta L < \pi$, $k_1 = k_2$ and (5.19) reduces to

$$\beta = (a \pm c)\left(k - \frac{k_1 + k_2}{2}\right) + b \tag{5.22}$$

This represents the simple crossing of the dispersion curves of two (uncoupled) modes.

6. RESONANT CAVITIES

6.1 Boundary conditions

A cavity is a volume of space enclosed by metallic walls, except for some holes which couple the cavity to the outer space. The metallic wall is very close to being an electric wall (or short circuit) with boundary conditions:

$$\left[\vec{n}\times\vec{E}\right]=0 \qquad \vec{n}\cdot\vec{H}=0$$

where $\vec{n}$ is a unit vector directed along the outward normal to the cavity surface. When $\omega \neq 0$, the condition on $\vec{H}$ is a consequence of the condition on $\vec{E}$ through Maxwell's equations (2.1).

In order to define an eigenmode of the cavity, one must also impose boundary conditions on a surface S' which closes the holes. There are two main possibilities:

1) The surface S' is also an electric wall (short circuit):

$$\left[\vec{n}\times\vec{E}\right]=0 \qquad \vec{n}\cdot\vec{H}=0 \qquad \text{on } S' \tag{6.1}$$

2) The surface S' is a magnetic wall (or open circuit):

$$\vec{n}\cdot\vec{E}=0 \qquad \left[\vec{n}\times\vec{H}\right]=0 \qquad \text{on } S' \,. \tag{6.2}$$

In the following, we only consider the first case, i.e. short-circuit modes. The open-circuit modes have different frequencies and different field patterns.

From Maxwell's equations at a given frequency, for a source free eigenmode, both $\vec{E}$ and $\vec{H}$ satisfy the homogeneous Helmholtz equation:

$$\Delta\vec{E}+k^2\vec{E}=0\ , \qquad \Delta\vec{H}+k^2\vec{H}=0\ . \tag{6.3}$$

These equations only hold when ε, μ are scalar quantities independent of position in space; this will always be assumed in the following. Moreover, for the fields $\vec{E}$ or $\vec{H}$ to be non-zero, k^2 may only take a discrete set of eigenvalues k_c^2.

In order to uniquely* determine a solution to this equation, *two* boundary conditions must be imposed on the whole surface S of the cavity ([8], p. 344). The closed surface S comprises the surface of the cavity walls and the surface S' which closes the holes. On S, one complements (6.1) with

$$\left[\vec{n}\times\vec{E}\right]=0 \quad \text{and} \quad \operatorname{div}\vec{E}=0 \qquad \text{for } \vec{E} \tag{6.4}$$

$$\vec{n}\cdot\vec{H}=0 \quad \text{and} \quad \left[\vec{n}\times\operatorname{curl}\vec{H}\right]=0 \qquad \text{for } \vec{H} \tag{6.5}$$

Suppose we have a solution $\vec{F}$ of the vector Helmholtz equation

$$\Delta\vec{F}+k_c^2\vec{F}=0\ . \tag{6.6}$$

Remembering that $\Delta \equiv \operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl}$, let us take curl and div of this equation; it follows

* Except in the degenerate cases, when several solutions have the same eigenvalue k_c^2 In these cases the general solution is a linear combination of appropriately chosen orthogonal basic solutions.

$$-\text{curl curl}\left(\text{curl } \vec{F}\right) + k_c^2\left(\text{curl } \vec{F}\right) = 0 \qquad \text{or} \qquad \Delta\left(\text{curl } \vec{F}\right) + k_c^2\left(\text{curl } \vec{F}\right) = 0$$

and

$$\Delta\left(\text{div } \vec{F}\right) + k_c^2\left(\text{div } \vec{F}\right) = 0 \tag{6.7}$$

hence

$$\text{grad div}\left(\text{grad div } \vec{F}\right) + k_c^2\left(\text{grad div } \vec{F}\right) = 0$$

or

$$\Delta\left(\text{grad div } \vec{F}\right) + k_c^2\left(\text{grad div } \vec{F}\right) = 0 \; .$$

Therefore, if we know a vector solution $\vec{F}$ of the Helmholtz equation (6.6), div $\vec{F}$ is a solution of the scalar Helmholtz equation (6.7), whereas curl $\vec{F}$ and grad div $\vec{F}$ are both solutions of the vector Helmholtz equation (6.6), all with the same eigenvalue k_c^2.

Suppose now we have an electric field solution $\vec{E}$ of the Helmholtz equation subject to the boundary conditions (6.4); then if curl $\vec{E} \neq 0$, it is a solution of the Helmholtz equation which on S satisfies the boundary conditions

$$\vec{n} \cdot \text{curl } \vec{E} = 0 \tag{6.8}$$

and

$$\left[\vec{n} \times \text{curl}\left(\text{curl } \vec{E}\right)\right] = \left[\vec{n} \times \text{grad div } \vec{E}\right] + k_c^2\left[\vec{n} \times \vec{E}\right] = 0 \; .$$

Indeed, the conditions (6.4) mean that the tangential component of $\vec{E}$, and div $\vec{E}$ vanish along the surface S; this, together with the Helmholtz equation, immediately entails (6.8). Comparing (6.8) with (6.5) we see that curl $\vec{E}$ satisfies the boundary conditions for $\vec{H}$.

If on the other hand div $\vec{E} \neq 0$, then grad div $\vec{E}$ is a solution of the Helmholtz equation which on S satisfies the boundary conditions

$$\left[\vec{n} \times \text{grad div } \vec{E}\right] = 0 \qquad \text{and} \qquad \text{div}\left(\text{grad div } \vec{E}\right) = -k_c^2 \text{ div } \vec{E} = 0 \; . \tag{6.9}$$

Both conditions follow from div $\vec{E} = 0$ on S. Comparing (6.9) with (6.4) we see that grad div $\vec{E}$ satisfies the same boundary conditions as $\vec{E}$.

Similarly, if we have a magnetic field solution $\vec{H}$ of the Helmholtz equation subject to the boundary conditions (6.5), curl $\vec{H}$ is a solution of the Helmholtz equation which on S satisfies the boundary conditions

$$\left[\vec{n} \times \text{curl } \vec{H}\right] = 0 \qquad \text{and} \qquad \text{div}\left(\text{curl } \vec{H}\right) = 0 \; . \tag{6.10}$$

Indeed, the first of these conditions follows from (6.5) while the second one is an identity. Comparing (6.10) with (6.4) we see that curl $\vec{H}$ satisfies the boundary conditions for $\vec{E}$.

If on the other hand div $\vec{H} \neq 0$, then grad div $\vec{H}$ is a solution of the Helmholtz equation which on S satisfies the boundary conditions

$$\vec{n} \cdot \text{grad div } \vec{H} = \vec{n} \cdot \text{curl curl } \vec{H} - k_c^2 \; \vec{n} \cdot \vec{H} = 0$$

(6.11)

and

$$\left[\vec{n} \times \text{curl}\left(\text{grad div } \vec{H}\right)\right] = 0 \; .$$

The first of these conditions follows from the Helmholtz equation and from (6.5), which entails that the tangential component of curl $\vec{H}$ vanishes along the surface S; the second condition is an identity. Comparing (6.11) with (6.5) we see that grad div $\vec{H}$ satisfies the same boundary conditions as $\vec{H}$.

Some properties of the eigenvalues k_c^2

For two arbitrary vector fields $\vec{F}$, $\vec{G}$ we have

$$\vec{G}\cdot\text{grad div }\vec{F} = \text{div}\left(\vec{G}\text{ div }\vec{F}\right) - \text{div }\vec{F}\cdot\text{div }\vec{G}$$

and
$$\vec{G}\cdot\text{curl curl }\vec{F} = \text{div}\left[\text{curl }\vec{F}\times\vec{G}\right] + \text{curl }\vec{F}\cdot\text{curl }\vec{G}\ .$$

Suppose, to be general, that $\vec{F}$ is a solution to the inhomogeneous Helmholtz equation

$$\Delta\vec{F} + k^2\vec{F} = \vec{K}\ . \tag{6.12}$$

By taking the dot product of this equation with $\vec{G}$ and using the above relations we obtain

$$\text{div}\left(\vec{G}\text{ div }\vec{F} + \left[\vec{G}\times\text{curl }\vec{F}\right]\right) - \text{div }\vec{F}\cdot\text{div }\vec{G} - \text{curl }\vec{F}\cdot\text{curl }\vec{G} + k^2\vec{F}\cdot\vec{G} = \vec{G}\cdot\vec{K}$$

hence

$$\oint_S\left(\vec{n}\cdot\vec{G}\text{ div }\vec{F} + \underbrace{\vec{n}\cdot\left[\vec{G}\times\text{curl }\vec{F}\right]}_{\text{curl }\vec{F}\cdot\left[\vec{n}\times\vec{G}\right] = -\vec{G}\cdot\left[\vec{n}\times\text{curl }\vec{F}\right]}\right)dS$$

$$-\int\left(\text{div }\vec{F}\cdot\text{div }\vec{G} + \text{curl }\vec{F}\cdot\text{curl }\vec{G}\right)dV + k^2\int\vec{F}\cdot\vec{G}\ dV = \int\vec{G}\cdot\vec{K}\ dV\ . \tag{6.13}$$

If we take $\vec{K} = 0$ (then $k^2 = k_c^2$) and $\vec{G} = \vec{F}^*$, either set of boundary conditions (6.4) or (6.5) for $\vec{F}$ makes the surface integral in (6.13) vanish; so we have

$$k_c^2\int\left|\vec{F}\right|^2 dV = \int\left(\left|\text{div }\vec{F}\right|^2 + \left|\text{curl }\vec{F}\right|^2\right)dV\ . \tag{6.14}$$

Therefore, the eigenvalue k_c^2 is a real number ≥ 0, and the eigenvector $\vec{F}$ may be taken as real. Moreover, $k_c^2 = 0$ is only possible when $\text{div }\vec{F} = 0$ and $\text{curl }\vec{F} = 0$.

In (6.13), $\vec{F}$ satisfies the inhomogeneous Helmholz equation (6.12) while $\vec{G}$ is arbitrary. Let us take $\vec{G} = \vec{F}_B$ where $\vec{F}_B$ satisfies the homogeneous Helmoltz equation

$$\Delta\vec{F}_B + k_B^2\vec{F}_B = 0\ .$$

Here k_B^2 represents the eigenvalue which corresponds to $\vec{F}_B$. Then there is a second relation similar to (6.13) where $\vec{K} = 0$, $\vec{F}$ is replaced by $\vec{F}_B^*$, k^2 is replaced by $k_B^{2*} = k_B^2$, and $\vec{G}$ is taken as $\vec{F}$. Taking the difference with (6.13) yields

$$\oint_S\left(\vec{n}\cdot\vec{F}_B^*\text{ div }\vec{F} - \vec{n}\cdot\vec{F}\text{ div }\vec{F}_B^* + \vec{n}\cdot\left[\vec{F}_B^*\times\text{curl }\vec{F}\right] - \vec{n}\cdot\left[\vec{F}\times\text{curl }\vec{F}_B^*\right]\right)dS$$

$$+\left(k^2 - k_B^2\right)\int\vec{F}\cdot\vec{F}_B^*\ dV = \int\vec{F}_B^*\cdot\vec{K}\ dV\ . \tag{6.15}$$

Note in passing that this relation is basic for the computation of the frequency shift due to any wall perturbation in a cavity ([20], p. 414).

Now we take $\vec{F} = \vec{F}_A$ where $\vec{F}_A$ satisfies the homogeneous Helmholtz equation

$$\Delta\vec{F}_A + k_A^2\vec{F}_A = 0\ .$$

Then in (6.15), $\vec{F}$ is $\vec{F}_A$, k^2 is k_A^2 and $\vec{K}$ is 0; hence

$$\oint_S \left(\vec{n}\cdot\vec{F}_B^* \operatorname{div} \vec{F}_A - \vec{n}\cdot\vec{F}_A \operatorname{div} \vec{F}_B^* + \vec{n}\cdot\left[\vec{F}_B^* \times \operatorname{curl} \vec{F}_A\right] - \vec{n}\cdot\left[\vec{F}_A \times \operatorname{curl} \vec{F}_B^*\right]\right) dS$$

$$+\left(k_A^2 - k_B^2\right) \int \vec{F}_A \cdot \vec{F}_B^* \, dV = 0 \ . \qquad (6.16)$$

If both $\vec{F}_A$, $\vec{F}_B$ satisfy the same set of boundary conditions on S, it being either (6.4) or (6.5), the surface integral in (6.16) vanishes; therefore the eigenmodes $\vec{F}_A$, $\vec{F}_B$ are orthogonal over the cavity volume when $A \neq B$.

6.2 Classification of modes

Starting from a vector solution $\vec{F}$ of the Helmholtz equation (6.6) subject to either boundary conditions (6.4) or (6.5), the derived vector fields curl curl $\vec{F}$ (when curl $\vec{F} \neq 0$) and grad div $\vec{F}$ (when div $\vec{F} \neq 0$) are both solutions of the Helmholtz equation with the same eigenvalue k_c^2; moreover they both satisfy the same boundary conditions as $\vec{F}$. We have to consider four possibilities ([26], p. 121 and p. 173):

1) curl $\vec{F} \neq 0$, div $\vec{F} \neq 0$

From (6.14) $k_c^2 \neq 0$; then the Helmholtz equation (6.6) may be written as

$$\vec{F} = \underbrace{\frac{1}{k_c^2}\operatorname{curl}\operatorname{curl} \vec{F}}_{\vec{F}_\ell} - \underbrace{\frac{1}{k_c^2}\operatorname{grad}\operatorname{div} \vec{F}}_{\vec{F}_\lambda} \qquad (6.17)$$

which shows that $\vec{F}$ is the sum of two parts $\vec{F}_\ell$ and $\vec{F}_\lambda$, each of which satisfies the Helmholtz equation with the same eigenvalue k_c^2, and also satisfies the same boundary conditions as $\vec{F}$. The first part $\vec{F}_\ell$ (solenoidal) has zero divergence; the second part $\vec{F}_\lambda$ (irrotational) has zero curl.

If the fields $\vec{F}$, $\vec{G}$ in (6.13) are taken as the solenoidal and the irrotational part of $\vec{F}$, since $\vec{K} = 0$ and $k^2 = k_c^2 \neq 0$, it follows immediately that these two parts are orthogonal over the cavity volume. Therefore the field $\vec{F}$ in (6.17) can be considered as the superposition of two orthogonal solutions to the Helmholtz equation, which are degenerate since they correspond to the same eigenvalue k_c^2.

When the degeneracy is removed (in most cases a slight deformation of the cavity wall is sufficient to achieve such an effect), the solution to the Helmholtz equation splits into a pure solenoidal field and a pure irrotational field having different eigenfrequencies. We are then led to three remaining possibilities.

2) curl $\vec{F} \neq 0$, div $\vec{F} = 0$

Assume that $\vec{F}$ satisfies the set of boundary conditions (6.4); we shall then write it as $\vec{E}_\ell$ (with an ordinary subscript ℓ for solenoidal modes). It has been shown in (6.8) that curl $\vec{E}_\ell$ is a solution of the Helmholtz equation with the same eigenvalue k_ℓ^2, which satisfies the set of boundary conditions (6.5); therefore we can let

$$\operatorname{curl} \vec{E}_\ell = a \vec{H}_\ell \quad \text{where } a \text{ is some scalar constant} \neq 0 \ . \qquad (6.18)$$

$\vec{H}_\ell$ is a solenoidal solution of the Helmholtz equation subject to the boundary conditions (6.5). Again, by (6.10) curl $\vec{H}_\ell$ is a solenoidal solution of the Helmholtz equation which satisfies the set of boundary conditions (6.4); therefore we anticipate that

$$\operatorname{curl} \vec{H}_\ell = b \vec{E}_\ell \quad \text{where } b \text{ is some scalar constant} \neq 0 \qquad (6.19)$$

From (6.18) and (6.19) we have

$$\operatorname{curl}\operatorname{curl} \vec{E}_\ell = ab \vec{E}_\ell \ .$$

With $$ab = k_\ell^2 \neq 0 \ , \tag{6.20}$$

this is precisely the Helmholtz equation (6.6) for $\vec{E}_\ell$.

The same result is obtained if one first assumes that $\vec{F}$ (= $\vec{H}_\ell$) satisfies the set of boundary conditions (6.5). Therefore, the solenoidal modes $\vec{E}_\ell$, $\vec{H}_\ell$ have the same eigenvalue k_ℓ^2 and are related by (6.18), (6.19) and (6.20). In particular, with $a = -j\omega_\ell\mu$ and $b = -j\omega_\ell\varepsilon$, they correspond to solutions of Maxwell's equations:

$$\text{curl } \vec{E}_\ell = -j\omega_\ell\mu \ \vec{H}_\ell \ , \qquad \text{curl } \vec{H}_\ell = j\omega_\ell\varepsilon \ \vec{E}_\ell \qquad \text{with} \qquad \omega_\ell^2\varepsilon\mu = k_\ell^2 > 0 \ . \tag{6.21}$$

Moreover, the volume integrals are linked by (6.14). Taking $\vec{F} = \vec{E}_\ell$ we obtain

$$k_\ell^2 \int |\vec{E}_\ell|^2 dV = \int |\text{curl } \vec{E}_\ell|^2 dV = \omega_\ell^2\mu^2 \int |\vec{H}_\ell|^2 dV$$

hence $$\varepsilon \int |\vec{E}_\ell|^2 dV = \mu \int |\vec{H}_\ell|^2 dV \ . \tag{6.22}$$

The same result is obtained by taking $\vec{F} = \vec{H}_\ell$. It simply expresses the equality (2.126) of time averaged electric and magnetic energies in a solenoidal mode.

3) curl $\vec{F} = 0$, div $\vec{F} \neq 0$

With $\vec{F}$ satisfying either set of boundary conditions (6.4) or (6.5), it has been shown in (6.9) and (6.11) that grad div $\vec{F}$ satisfies the same set of boundary conditions as $\vec{F}$. In fact, for curl $\vec{F} = 0$ the Helmholtz equation

$$\text{grad div } \vec{F} + k_\lambda^2\vec{F} = 0 \qquad \text{where from (6.14)} \quad k_\lambda^2 \neq 0 \tag{6.23}$$

shows that $\vec{F}$ and grad div $\vec{F}$ are the same vector fields, up to a scalar multiplier $(-k_\lambda^2)$.

When $\vec{F}$ satisfies the set of boundary conditions (6.4), we shall designate it by $\vec{E}_\lambda$ (with a Greek subscript λ for irrotational modes). Since curl $\vec{E}_\lambda = 0$, we may write

$$\vec{E}_\lambda = \text{grad } \phi_\lambda \ . \tag{6.24}$$

Inserting (6.24) into (6.23) yields

$$\text{grad}\left(\Delta\phi_\lambda + k_\lambda^2\phi_\lambda\right) = 0 \qquad \text{or} \qquad \Delta\phi_\lambda + k_\lambda^2\phi_\lambda = \text{ constant.}$$

Since ϕ_λ is only determined up to an additive constant and since $k_\lambda^2 \neq 0$, one can always choose the additive constant such that

$$\Delta\phi_\lambda + k_\lambda^2\phi_\lambda = 0, \qquad k_\lambda^2 \neq 0. \tag{6.25}$$

The boundary conditions (6.4) require that on S, ϕ_λ = constant and $\Delta\phi_\lambda = 0$. With (6.25), both conditions are satisfied when

$$\phi_\lambda = 0 \qquad \text{on } S \ . \tag{6.26}$$

From (6.23) divided by $(-k_\lambda^2)$ and (6.24) it appears that one may take

$$\phi_\lambda = -\frac{1}{k_\lambda^2} \text{ div } \vec{E}_\lambda \ .$$

Equation (6.7) shows that this determination of ϕ_λ satisfies (6.25); from (6.4) it is obvious that it also satisfies (6.26). Multiplying (6.25) by ϕ_λ^* and integrating over the cavity volume yields

$$\oint \phi_\lambda^* \frac{\partial \phi_\lambda}{\partial n} dS - \int |\vec{E}_\lambda|^2 dV + k_\lambda^2 \int |\phi_\lambda|^2 dV = 0 . \tag{6.27}$$

With (6.26) this confirms that k_λ^2 is real and positive.

When $\vec{F}$ satisfies the set of boundary conditions (6.5), we shall designate it by $\vec{H}_\lambda$. Again we may write

$$\vec{H}_\lambda = \text{grad } \psi_\lambda \tag{6.28}$$

with
$$\Delta\psi_\lambda + k_\lambda^2 \psi_\lambda = 0, \qquad k_\lambda^2 \neq 0 . \tag{6.29}$$

The boundary conditions (6.5) require that

$$\frac{\partial \psi_\lambda}{\partial n} = 0 \qquad \text{on } S . \tag{6.30}$$

Again, one may take
$$\psi_\lambda = -\frac{1}{k_\lambda^2} \text{ div } \vec{H}_\lambda;$$

that it satisfies (6.30) is obvious from (6.5) because (6.23) entails

$$\text{grad}\left(-\frac{1}{k_\lambda^2} \text{div} \vec{H}_\lambda\right) = \vec{H}_\lambda .$$

Similarly, from (6.29) we deduce

$$\oint \psi_\lambda^* \frac{\partial \psi_\lambda}{\partial n} dS - \int |\vec{H}_\lambda|^2 dV + k_\lambda^2 \int |\psi_\lambda|^2 dV = 0 . \tag{6.31}$$

From (6.30) it again follows that k_λ^2 is real and positive.

Remarks

a) Since the boundary conditions (6.26) and (6.30) are different for $\vec{E}_\lambda$ and $\vec{H}_\lambda$, the eigenvalues k_λ^2 are in general different; therefore there is no relation between $\vec{E}_\lambda$ and $\vec{H}_\lambda$. Physically, $\vec{E}_\lambda$ (or $\vec{H}_\lambda$) is a static field ($\omega = 0$) produced by surface charges (or currents) such that the boundary conditions (6.4) or (6.5) are fulfilled. When $k_\lambda^2 \neq 0$, div $\vec{F} \neq 0$: the solution of the Helmholtz equation (6.23) also requires the existence of a volume distribution of electric (or magnetic) charges.

b) The solutions of the scalar Helmholtz equation (6.25) or (6.29) subject to the boundary conditions (6.26) or (6.30), form a complete orthogonal basis for expanding a scalar function in the cavity volume. In the case of Neumann's boundary condition (6.30), to the eigenvalue $k_\lambda^2 = 0$ there corresponds a non-zero eigenfunction ψ_λ = constant, which *must* be included in the set of eigenfunctions in order to have it complete.

4) $\text{curl} \vec{F} = 0, \quad \text{div} \vec{F} = 0$

From (6.14) this is the only case where the eigenvalue k_λ^2 is zero. Such modes could be considered to be either solenoidal or irrotational; but since they show no relation between $\vec{E}$ and $\vec{H}$ type of modes, we shall consider them mainly as irrotational and designate them with a Greek subscript. Because their eigenvalue $k_\lambda^2 = 0$ is the smallest possible, we shall number them with $\lambda = 0$. On the other hand, these modes might equally well be considered as solenoidal with $k_l^2 = 0$.

From curl $\vec{F} = 0$ one can still write (6.24) or (6.28); the condition div $\vec{F} = 0$ still takes the form of (6.25) or (6.29) where now $k_\lambda^2 = 0$.

Types of fields for $k_\lambda^2 = 0$

When $\vec{F}$ satisfies the set of boundary conditions (6.4), one must have

$$\phi_0 = \text{constant } \phi_{0i} \quad \text{on each disconnected part } S_i \text{ of } S \tag{6.32}$$

With $k_\lambda^2 = 0$, (6.25) entails

$$\sum_i \oint_{S_i} \frac{\partial \phi_0}{\partial n}\, dS = 0 \tag{6.33}$$

whereas (6.27) yields

$$\sum_i \phi_{0i}^* \oint_{S_i} \frac{\partial \phi_0}{\partial n} dS = \int | \vec{E}_0 |^2 dV \tag{6.34}$$

If the surface S is in one piece, (6.34) together with (6.33) implies $\vec{E}_0 = 0$. If the surface S is made up of several disconnected parts, ϕ_0 can take different ϕ_{0i} values on the different parts, and $\vec{E}_0$ is the electrostatic field between several unconnected conductors which are raised to different potentials (see Fig. 6.1b).

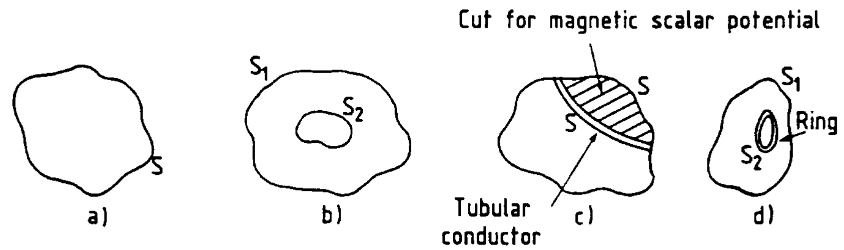


Fig. 6.1 The four basic types of cavities:

a) simply connected, with a single surface ($\vec{E}_0 = 0$, $\vec{H}_0 = 0$)

b) simply connected, with a surface in several parts ($\vec{E}_0 \neq 0$, $\vec{H}_0 = 0$)

c) multiply connected, with a single surface ($\vec{E}_0 = 0$, $\vec{H}_0 \neq 0$)

d) multiply connected, with a surface in several parts ($\vec{E}_0 \neq 0$, $\vec{H}_0 \neq 0$).

When $\vec{F}$ satisfies the set of boundary conditions (6.5), ψ_0 still has to satisfy (6.30) on S. If ψ_0 is univalued, (6.30) together with (6.31) where $k_\lambda^2 = 0$, implies $\vec{H}_0 = 0$. To have $\vec{H}_0 \neq 0$ requires ψ_0 to be multivalued, which is possible only if a current (and therefore a metallic conductor) makes a loop inside the cavity. By introducing a cut inside the cavity volume, in the form of a surface having the loop and the cavity walls as boundaries, such that it prevents a full turn around the metallic conductor (see Fig. 6.1c), one makes ψ_0 univalued but taking different values ψ_{0+} and ψ_{0-} on both sides of the cut. Since $\vec{H}_0 = \text{grad}\, \psi_0$ is continuous across the cut, the difference $(\psi_{0+} - \psi_{0-})$ must be a constant along the cut. With (6.30), the only surface integrals which are left in (6.31) are the integrals on both sides of the cut; therefore

$$\int_{cut} \left(\psi_{0+}^* - \psi_{0-}^*\right) \frac{\partial \psi_0}{\partial n_+}\, dS = \left(\psi_{0+}^* - \psi_{0-}^*\right) \int_{cut} \underbrace{\frac{\partial \psi_0}{\partial n_+} dS}_{\text{flux of the magnetic field through the cut}} = \int | \vec{H}_0 |^2 dV. \tag{6.35}$$

$\vec{H}_0$ is then the D.C. magnetic field produced by D.C. surface currents on the cavity walls, such that $\vec{n} \cdot \vec{H}_0 = 0$ on the walls (which corresponds to superconducting walls).

6.3 Expansion of the electromagnetic field inside a cavity

Once all the solenoidal modes $\vec{E}_\ell, \vec{H}_\ell$ and the irrotational modes $\vec{E}_\lambda$, $\vec{H}_\lambda$ of a cavity have been determined, one may represent the fields in the cavity as

$$\vec{E} = \sum_\ell a_\ell \vec{E}_\ell + \sum_\lambda a_\lambda \vec{E}_\lambda \qquad \text{and} \qquad \vec{H} = \sum_\ell b_\ell \vec{H}_\ell + \sum_\lambda b_\lambda \vec{H}_\lambda \tag{6.36}$$

where different $\vec{E}$ (or $\vec{H}$) modes are orthogonal over the cavity volume, i.e.

$$\int \vec{E}_A \cdot \vec{E}_B^* \; dV = 0 \qquad \text{and} \qquad \int \vec{H}_A \cdot \vec{H}_B^* \; dV = 0 \qquad \text{when } A \neq B\,. \tag{6.37}$$

Here A and B represent any kind of subscripts, Latin or Greek.

If A and B modes are both either solenoidal or irrotational, (6.37) follows from (6.16). If one mode is solenoidal and the other is irrotational, (6.37) follows from (6.13) where k^2 is taken either as k_A^2 or as k_B^2, provided one of them is different from zero; but (6.13) ensures the orthogonality of a solenoidal and an irrotational mode even when $k_A^2 = k_B^2 \neq 0$. The fact that irrotational modes are orthogonal to solenoidal modes proves that no one of these sets of modes is complete; both are needed to form a complete basis of vector functions.

Since the eigenvalues k_A^2 of the Helmholtz equation (6.6) and the boundary conditions (6.4) or (6.5) are real, the eigenvectors $\vec{E}_A$ or $\vec{H}_A$ can be taken as real except for some constant complex factor. Therefore the orthogonality relations (6.37) may be written with or without an asterisk; the advantage of writing them with an asterisk is that when $A = B$, the integrals in (6.37) are real and positive.

From (6.37) it follows immediately that

$$a_B = \frac{\int \vec{E} \cdot \vec{E}_B^* \; dV}{\int |\vec{E}_B|^2 dV} \qquad\qquad b_B = \frac{\int \vec{H} \cdot \vec{H}_B^* \; dV}{\int |\vec{H}_B|^2 dV}\,. \tag{6.38}$$

One of the most illuminating ways of deriving the coefficients (6.38) is to use Maxwell's equations written in a completely symmetrical form with respect to electric and magnetic quantities ([33], p. 191):

$$\text{curl } \vec{H} = \vec{J} + j\omega\varepsilon \; \vec{E} \tag{6.39}$$

$$\text{curl } \vec{E} = -\vec{J}_m - j\omega\mu \; \vec{H}\,. \tag{6.40}$$

In (6.40) $\vec{J}_m$ is a fictitious magnetic current density, which ultimately will be made equal to zero. The electric current $\vec{J}$ will be expanded by using the same base vectors as $\vec{E}$, while the magnetic current $\vec{J}_m$ will be expanded by using the same base vectors as $\vec{H}$. In the following, ε and μ are assumed to be real; then, with (6.21), ω_ℓ is also real.

Expansion of the electric current density

From (6.39) we have

$$\int \vec{J} \cdot \vec{E}_A^* dV = \int \left(\text{curl } \vec{H} - j\omega\varepsilon \; \vec{E}\right) \cdot \vec{E}_A^* dV = \int \left(\text{div}\left[\vec{H} \times \vec{E}_A^*\right] + \vec{H} \cdot \text{curl } \vec{E}_A^*\right) dV - j\omega\varepsilon \int \vec{E} \cdot \vec{E}_A^* dV$$

or

$$\int \vec{J} \cdot \vec{E}_A^* dV - \oint \left[\vec{n} \times \vec{H}\right] \cdot \vec{E}_A^* dS = \int \vec{H} \cdot \text{curl } \vec{E}_A^* dV - j\omega\varepsilon \int \vec{E} \cdot \vec{E}_A^* dV \tag{6.41}$$

The left-hand side of (6.41) can be considered as the scalar product of $\vec{E}_A^*$ with an electric current volume density

$$\vec{J}_{total} = \vec{J} - \left[\; \vec{n} \times \vec{H}\right] \cdot \delta(n) \tag{6.42}$$

where n is the distance from the cavity surface, measured along the normal to the surface; $-[\vec{n} \times \vec{H}]$ would be the electric current surface density if $\vec{H} = 0$ outside the surface (i.e. inside the metal). Because of the boundary conditions (6.4) for $\vec{E}_A$, the surface integral in (6.41) vanishes.

For the solenoidal modes, (6.41) becomes, using (6.21):

$$\int \vec{J} \cdot \vec{E}_\ell^* dV = -j\omega\varepsilon \int \vec{E} \cdot \vec{E}_\ell^* dV + j\omega_\ell \mu \int \vec{H} \cdot \vec{H}_\ell^* dV. \tag{6.43}$$

For the irrotational modes, (6.41) simply reads

$$\int \vec{J} \cdot \vec{E}_\lambda^* dV = -j\omega\varepsilon \int \vec{E} \cdot \vec{E}_\lambda^* dV. \tag{6.44}$$

Expansion of the magnetic current density

From (6.40) we have

$$\int \vec{J}_m \cdot \vec{H}_A^* dV = \int \left(-\text{curl } \vec{E} - j\omega\mu \ \vec{H}\right) \cdot \vec{H}_A^* dV = \int \left(-\text{div}\left[\vec{E} \times \vec{H}_A^*\right] - \vec{E} \cdot \text{curl } \vec{H}_A^*\right) dV - j\omega\mu \int \vec{H} \cdot \vec{H}_A^* dV$$

or
$$\int \vec{J}_m \cdot \vec{H}_A^* dV + \oint \left[\vec{n} \times \vec{E}\right] \cdot \vec{H}_A^* dS = -\int \vec{E} \cdot \text{curl } \vec{H}_A^* dV - j\omega\mu \int \vec{H} \cdot \vec{H}_A^* dV. \tag{6.45}$$

The left-hand side of (6.45) can be considered as the scalar product of $\vec{H}_A^*$ with a magnetic current volume density

$$\vec{J}_{m \ total} = \vec{J}_m + \left[\vec{n} \times \vec{E}\right] \cdot \delta(n) \tag{6.46}$$

where $[\vec{n} \times \vec{E}]$ would be the magnetic current surface density if $\vec{E} = 0$ outside the surface (i.e. inside the metal). With the boundary conditions (6.5) for $\vec{H}_A$, the surface integral in (6.45) does not vanish.

For the solenoidal modes, (6.45) becomes with (6.21):

$$\int \vec{J}_m \cdot \vec{H}_\ell^* dV + \oint \left[\vec{n} \times \vec{E}\right] \cdot \vec{H}_\ell^* dS = j\omega_\ell \varepsilon \int \vec{E} \cdot \vec{E}_\ell^* dV - j\omega\mu \int \vec{H} \cdot \vec{H}_\ell^* dV. \tag{6.47}$$

For the irrotational modes, (6.45) simply reads

$$\int \vec{J}_m \cdot \vec{H}_\lambda^* dV + \oint \left[\vec{n} \times \vec{E}\right] \cdot \vec{H}_\lambda^* dS = -j\omega\mu \int \vec{H} \cdot \vec{H}_\lambda^* dV. \tag{6.48}$$

Solenoidal mode coefficients a_ℓ, b_ℓ

They can be computed from the system of equations (6.43), (6.47); the result is

$$\left(k^2 - k_\ell^2\right) \int \vec{E} \cdot \vec{E}_\ell^* dV = j\omega\mu \int \vec{J} \cdot \vec{E}_\ell^* dV + j\omega_\ell \mu \left(\int \vec{J}_m \cdot \vec{H}_\ell^* dV + \oint \left[\vec{n} \times \vec{E}\right] \cdot \vec{H}_\ell^* dS\right) \tag{6.49a}$$

$$\left(k^2 - k_\ell^2\right) \int \vec{H} \cdot \vec{H}_\ell^* dV = j\omega_\ell \varepsilon \int \vec{J} \cdot \vec{E}_\ell^* dV + j\omega\varepsilon \left(\int \vec{J}_m \cdot \vec{H}_\ell^* dV + \oint \left[\vec{n} \times \vec{E}\right] \cdot \vec{H}_\ell^* dS\right) \tag{6.49b}$$

Irrotational mode coefficients a_λ, b_λ

They are given straightforwardly by (6.44) and (6.48):

$$\int \vec{E} \cdot \vec{E}_\lambda^* dV = -\frac{1}{j\omega\varepsilon} \int \vec{J} \cdot \vec{E}_\lambda^* dV \tag{6.50a}$$

$$\int \vec{H}\cdot\vec{H}_\lambda^* dV = -\frac{1}{j\omega\mu}\left(\int \vec{J}_m\cdot\vec{H}_\lambda^* dV + \oint \left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_\lambda^* dS\right) . \tag{6.50b}$$

Using (6.49) and (6.50) yields the a_B and b_B coefficients through (6.38). When remembering (6.22), the resulting expressions (6.36) for $\vec{E}$ and $\vec{H}$ agree with those given by Van Bladel ([33], p. 299).

The irrotational $\vec{E}_\lambda$ modes with $\lambda \neq 0$, having $\operatorname{div} \vec{E}_\lambda \neq 0$, are necessary in order to represent a non-zero $\varepsilon \operatorname{div} \vec{E} = \rho$, which occurs if there are electric charges in the cavity. Similarly, the irrotational $\vec{H}_\lambda$ modes with $\lambda \neq 0$, having $\operatorname{div} \vec{H}_\lambda \neq 0$, are necessary in order to represent a non-zero $\mu \operatorname{div} \vec{H} = \rho_m$. In fact, although there is no volume density ρ_m of magnetic charge inside the cavity, there is a fictitious surface density $\sigma_m = -\mu \vec{H}\cdot\vec{n}$ associated with the fictitious magnetic current surface density $[\vec{n} \times \vec{E}]$ ([34], p. 181).

Remark: It is interesting to verify directly that the expressions (6.36) for $\vec{E}$ and $\vec{H}$ do satisfy Maxwell's equations (6.39) and (6.40).

From (6.21):

$$\operatorname{curl}\vec{H} - j\omega\varepsilon\,\vec{E} = \sum_\ell (j\omega_\ell\varepsilon b_\ell - j\omega\varepsilon a_\ell)\vec{E}_\ell - j\omega\varepsilon\sum_\lambda a_\lambda \vec{E}_\lambda .$$

With (6.22), (6.38) and (6.43), (6.44) this expression transforms into

$$\operatorname{curl}\,\vec{H} - j\omega\varepsilon\,\vec{E} = \sum_\ell \vec{E}_\ell \frac{\int \vec{J}\cdot\vec{E}_\ell^* dV}{\int \left|\vec{E}_\ell\right|^2 dV} + \sum_\lambda \vec{E}_\lambda \frac{\int \vec{J}\cdot\vec{E}_\lambda^* dV}{\int \left|\vec{E}_\lambda\right|^2 dV} = \vec{J} \quad \text{inside } V . \tag{6.51}$$

This is Maxwell's equation (6.39).

Similarly, from (6.21):

$$-\operatorname{curl}\,\vec{E} - j\omega\mu\,\vec{H} = \sum_\ell (j\omega_\ell\mu a_\ell - j\omega\mu b_\ell)\vec{H}_\ell - j\omega\mu\sum_\lambda b_\lambda \vec{H}_\lambda .$$

With (6.22), (6.38) and (6.47), (6.48) this expression transforms into

$$-\operatorname{curl}\,\vec{E} - j\omega\mu\,\vec{H} = \sum_\ell \vec{H}_\ell \frac{\int \vec{J}_m\cdot\vec{H}_\ell^* dV + \oint\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_\ell^* dS}{\int \left|\vec{H}_\ell\right|^2 dV} + \sum_\lambda \vec{H}_\lambda \frac{\int \vec{J}_m\cdot\vec{H}_\lambda^* dV + \oint\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_\lambda^* dS}{\int \left|\vec{H}_\lambda\right|^2 dV}$$
$$= \vec{J}_m + \left[\vec{n}\times\vec{E}\right]\cdot\delta(n) .$$

This is Maxwell's equation (6.40) except for the last term in (6.52). Since this term is zero inside V, Maxwell's equation (6.40) is again satisfied *inside* V; but the $\delta(n)$ term is responsible for producing non-zero coefficients in its own expansion.

Effect of a finite wall impedance

In (6.49), the surface integral prevents the coefficients a_ℓ, b_ℓ from becoming infinite when $k^2 = k_\ell^2$. In both (6.49) and (6.50b), the surface integral can be split as

$$\oint_S \left(\vec{n}\times\vec{E}\right)\cdot\vec{H}_A^* dS = \int_{\text{holes}} \left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_A^* dS + \int_{\text{walls}} \left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_A^* dS . \tag{6.53}$$

On the metallic walls of the cavity, we apply the approximate boundary condition (3.24):

$$\vec{E}_t = Z_s\left[\vec{H}_t\times\vec{n}\right] \quad \text{or} \quad \left[\vec{n}\times\vec{E}\right] = Z_s\vec{H}_t .$$

Since $\vec{n}\cdot\vec{H}_A = 0$ on S, the last integral in (6.53) becomes, with (6.36):

$$\int_{\text{walls}}\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_A^* dS = \int_{\text{walls}} Z_s\vec{H}\cdot\vec{H}_A^* dS = \sum_m b_m \int_{\text{walls}} Z_s\vec{H}_m\cdot\vec{H}_A^* dS + \sum_\nu b_\nu \int_{\text{walls}} Z_s\vec{H}_\nu\cdot\vec{H}_A^* dS \tag{6.54}$$

Introducing (6.53), (6.54) in (6.49b) and (6.50b), while remembering (6.38) yields

$$\left(k^2-k_\ell^2\right)b_\ell\int\left|\vec{H}_\ell\right|^2 dV - j\omega\varepsilon\sum_m b_m \int_{\text{walls}} Z_s\vec{H}_m\cdot\vec{H}_\ell^* dS - j\omega\varepsilon\sum_\nu b_\nu \int_{\text{walls}} Z_s\vec{H}_\nu\cdot\vec{H}_\ell^* dS$$

$$= j\omega_\ell\varepsilon\int\vec{J}\cdot\vec{E}_\ell^* dV + j\omega\varepsilon\left[\int\vec{J}_m\cdot\vec{H}_\ell^* dV + \int_{\text{holes}}\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_\ell^* dS\right] \tag{6.55}$$

and

$$b_\lambda\int\left|\vec{H}_\lambda\right|^2 dV + \frac{1}{j\omega\mu}\sum_m b_m \int_{\text{walls}} Z_s\vec{H}_m\cdot\vec{H}_\lambda^* dS + \frac{1}{j\omega\mu}\sum_\nu b_\nu \int_{\text{walls}} Z_s\vec{H}_\nu\cdot\vec{H}_\lambda^* dS$$

$$= -\frac{1}{j\omega\mu}\left[\int\vec{J}_m\cdot\vec{H}_\lambda^* dV + \int_{\text{holes}}\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_\lambda^* dS\right]. \tag{6.56}$$

The set (6.55), (6.56) constitutes an infinite system of linear equations for the b_ℓ's and the b_λ's. In order to obtain a simple approximate solution to this system, we shall only keep in (6.55) the diagonal term $m = \ell$, and in (6.56) the diagonal term $\nu = \lambda$, because the diagonal terms are likely to be the most important ones. Then (6.55) reduces to

$$\left[\left(k^2-k_\ell^2\right)\int\left|\vec{H}_\ell\right|^2 dV - j\omega\varepsilon\int_{\text{walls}} Z_s\left|\vec{H}_\ell\right|^2 dS\right]b_\ell$$

$$= j\omega_\ell\varepsilon\int\vec{J}\cdot\vec{E}_\ell^* dV + j\omega\varepsilon\left[\int\vec{J}_m\cdot\vec{H}_\ell^* dV + \int_{\text{holes}}\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_\ell^* dS\right] \tag{6.57}$$

while (6.56) reduces to

$$\left[\int\left|\vec{H}_\lambda\right|^2 dV + \frac{1}{j\omega\mu}\int_{\text{walls}} Z_s\left|\vec{H}_\lambda\right|^2 dS\right]b_\lambda = -\frac{1}{j\omega\mu}\left[\int\vec{J}_m\cdot\vec{H}_\lambda^* dV + \int_{\text{holes}}\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_\lambda^* dS\right]. \tag{6.58}$$

With (6.22) and the definition (2.127) of Q we have, for a solenoidal mode ℓ:

$$\frac{|\omega_\ell|}{Q_\ell} = \frac{\int_{\text{walls}} R_s(\omega_\ell)\left|\vec{H}_\ell\right|^2 dS}{\int\mu\left|\vec{H}_\ell\right|^2 dV}. \tag{6.59}$$

If we define the Q factor of an irrotational mode λ by the same formula (6.59) with ℓ replaced by λ, we shall have in both cases, using (3.25):

$$\int_{\text{walls}} Z_s(\omega)\left|\vec{H}_A\right|^2 dS = [1+j\ \mathrm{sgn}(\omega)]\sqrt{\left|\frac{\omega}{\omega_A}\right|}\int_{\text{walls}} R_s(\omega_A)\left|\vec{H}_A\right|^2 dS = \frac{1+j\ \mathrm{sgn}(\omega)}{Q_A}\sqrt{|\omega\omega_A|}\ \mu\int\left|\vec{H}_A\right|^2 dV\ . \tag{6.60}$$

It should be noticed that even when $\omega_\lambda = 0$, the ratio $\sqrt{|\omega_\lambda|}/Q_\lambda$ which appears in (6.60) is still perfectly defined by (6.59).

Therefore (6.57) and (6.58) becomes, with (6.38):

$$\left[k^2 - k_\ell^2 + k^2 \frac{1-j\ \mathrm{sgn}(\omega)}{Q_\ell}\sqrt{\left|\frac{\omega_\ell}{\omega}\right|}\right]\int \vec{H}\cdot\vec{H}_\ell^* dV = j\omega_\ell\varepsilon\int \vec{J}\cdot\vec{E}_\ell^* dV + j\omega\varepsilon\left[\int \vec{J}_m\cdot\vec{H}_\ell^* dV + \int_{\text{holes}}\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_\ell^* dS\right] \tag{6.61}$$

and
$$\left[1+\frac{1-j\ \mathrm{sgn}(\omega)}{Q_\lambda}\sqrt{\left|\frac{\omega_\lambda}{\omega}\right|}\right]\int \vec{H}\cdot\vec{H}_\lambda^* dV = -\frac{1}{j\omega\mu}\left[\int \vec{J}_m\cdot\vec{H}_\lambda^* dV + \int_{\text{holes}}\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_\ell^* dS\right]. \tag{6.62}$$

Finally, the solenoidal mode coefficients for the electric field are readily obtained by combining (6.43) and (6.61):

$$\left[k^2 - k_\ell^2 + k^2 \frac{1-j\ \mathrm{sgn}(\omega)}{Q_\ell}\sqrt{\left|\frac{\omega_\ell}{\omega}\right|}\right]\int \vec{E}\cdot\vec{E}_\ell^* dV$$

$$= j\omega\mu\left[1+\frac{1-j\ \mathrm{sgn}(\omega)}{Q_\ell}\sqrt{\left|\frac{\omega_\ell}{\omega}\right|}\right]\int \vec{J}\cdot\vec{E}_\ell^* dV + j\omega_\ell\mu\left[\int \vec{J}_m\cdot\vec{H}_\ell^* dV + \int_{\text{holes}}\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_\ell^* dS\right] \tag{6.63}$$

whereas the irrotational mode coefficients for the electric field are still given directly by (6.44):

$$\int \vec{E}\cdot\vec{E}_\lambda^* dV = -\frac{1}{j\omega\varepsilon}\int \vec{J}\cdot\vec{E}_\lambda^* dV\ . \tag{6.64}$$

Remark: From the derivation above, it is clear that the expressions (6.61) to (6.63) are only a first approximation; but it is very difficult to go beyond this approximation when taking the wall losses into account. This approximation is essentially valid in the vicinity of each solenoidal mode ω_ℓ.

6.4 Excitation of cavities

A cavity can be excited:

a) by an electric current inside the cavity. The current may be carried by a probe, a loop, or it may be a convection current.

b) by a tangential $\vec{E}$ on the surface of the holes (which is equivalent to a surface magnetic current).

Using (6.61) to (6.64) in (6.36) and (6.38), we can write the expressions for the phasors $\vec{E}(\omega)$ and $\vec{H}(\omega)$:

$$\vec{E}(\omega) = -\sum_\nu \frac{\vec{E}_\nu}{j\omega\varepsilon}\frac{\int \vec{J}\cdot\vec{E}_\nu^* dV}{\int\left|\vec{E}_\nu\right|^2 dV} - \sum_n \vec{E}_n \frac{j\omega\mu\left(1+\frac{1-j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\left|\frac{\omega_n}{\omega}\right|}\right)}{k_n^2 - k^2\left(1+\frac{1-j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\left|\frac{\omega_n}{\omega}\right|}\right)}\frac{\int \vec{J}\cdot\vec{E}_n^* dV}{\int\left|\vec{E}_n\right|^2 dV}$$

$$-\sum_n \vec{E}_n \frac{j\omega_n\mu}{k_n^2 - k^2\left(1+\frac{1-j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\left|\frac{\omega_n}{\omega}\right|}\right)}\frac{\int_{\text{holes}}\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_n^* dS + \int \vec{J}_m\cdot\vec{H}_n^* dV}{\int\left|\vec{E}_n\right|^2 dV} \tag{6.65}$$

$$\vec{H}(\omega) = -\sum_{\nu} \frac{\vec{H}_\nu}{j\omega\mu\left(1+\frac{1-j\ \mathrm{sgn}(\omega)}{Q_\nu}\sqrt{\left|\frac{\omega_\nu}{\omega}\right|}\right)} \frac{\int_{\text{holes}}\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_\nu^* dS + \int \vec{J}_m \cdot \vec{H}_\nu^*\ dV}{\int\left|\vec{H}_\nu\right|^2 dV}$$

$$- \sum_n \vec{H}_n \frac{j\omega_n\varepsilon}{k_n^2 - k^2\left(1+\frac{1-j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\left|\frac{\omega_n}{\omega}\right|}\right)} \frac{\int \vec{J}\cdot\vec{E}_n^* dV}{\int\left|\vec{H}_n\right|^2 dV}$$

$$- \sum_n \vec{H}_n \frac{j\omega\varepsilon}{k_n^2 - k^2\left(1+\frac{1-j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\left|\frac{\omega_n}{\omega}\right|}\right)} \frac{\int_{\text{holes}}\left[\vec{n}\times\vec{E}\right]\cdot\vec{H}_n^* dS + \int \vec{J}_m \cdot \vec{H}_n^*\ dV}{\int\left|\vec{H}_n\right|^2 dV}. \tag{6.66}$$

In these expressions we have used the equality (6.22) between electric and magnetic stored energies for solenoidal modes.

The rigorous expansion coefficients (6.49), (6.50) would yield expressions similar to (6.65), (6.66), except that all brackets containing Q-factors would be replaced by 1 (there would be no terms containing Q-factors), and that the surface integrals, instead of being restricted to the holes, would be extended over the whole surface S enclosing the cavity.

Excitation of cavities by holes has been treated by Kurokawa [26, 34]. In what follows we shall only consider excitation by electric currents. The terms corresponding to current excitation in (6.65) and (6.66) are essentially the same as given by Collin ([8], p. 359) and Van Bladel ([33], p. 299).

The need for the irrotational mode terms has been emphasized many times in the literature. Their meaning in the particular case of a rectangular cavity has been discussed by Schelkunoff [35]; it has also been shown by measurements that they are necessary for representing the fields in a rectangular cavity excited by a probe, especially at frequencies below the first resonance of the cavity [36].

Frequency behaviour of the resonant terms in (6.65)

The last terms in (6.65) behave like

$$\frac{j\omega\mu\left(1+\frac{1-j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\left|\frac{\omega_n}{\omega}\right|}\right)}{k_n^2 - k^2\left(1+\frac{1-j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\left|\frac{\omega_n}{\omega}\right|}\right)} = \frac{1}{\dfrac{k_n^2}{j\omega\mu\left(1+\frac{1-j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\left|\frac{\omega_n}{\omega}\right|}\right)} + j\omega\varepsilon}$$

$$= \frac{1}{k_n\left[\dfrac{k_n}{\left(j\omega + \frac{1+j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\left|\omega\ \omega_n\right|}\right)\mu} + \dfrac{j\omega\varepsilon}{k_n}\right]}. \tag{6.67}$$

The square bracket in the last expression represents the admittance of a parallel resonant circuit whose two branches are a capacitance ε/k_n, and an inductance μ/k_n having an internal impedance

$$\frac{1+j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\left|\omega\ \omega_n\right|}\ \mu / k_n$$

which is proportional to the surface impedance $Z_s(\omega)$ of the cavity walls.

In (6.67), the

$$\frac{1-j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\frac{|\omega_n|}{|\omega|}}$$

term is a small correction which is only important in the vicinity of the resonant frequency ω_n; therefore, we shall replace it by $[1-j\ \mathrm{sgn}(\omega)]/Q_n$. The inverse of (6.67) then becomes

$$\frac{k_n^2}{j\omega\mu\left(1+\dfrac{1-j\ \mathrm{sgn}(\omega)}{Q_n}\right)}+j\omega\varepsilon \approx \frac{k_n^2}{j\omega\mu\left(1+\dfrac{1}{Q_n}\right)\left(1-\dfrac{j\ \mathrm{sgn}(\omega)}{Q_n}\right)}+j\omega\varepsilon = \frac{k_n'^2}{j\omega\mu\left(1-\dfrac{j\ \mathrm{sgn}(\omega)}{Q_n}\right)}+j\omega\varepsilon$$

$$\approx \frac{k_n'^2}{|\omega|\mu Q_n}+\frac{k_n'^2}{j\omega\mu}+j\omega\varepsilon \tag{6.68}$$

where

$$k_n'^2=\omega_n'^2\varepsilon\mu=\frac{k_n^2}{1+\dfrac{1}{Q_n}} \tag{6.69}$$

and ω_n' is the n^{th} resonant frequency of the cavity, as corrected for the finite skin depth.

Again, in (6.68), the real part $k_n'^2/(|\omega|\mu Q_n)$ is only important when ω is close to ω_n'; therefore we shall replace it by its value for $\omega=\omega_n$. By using (6.69), the inverse of (6.67) is transformed approximately into

$$\frac{|\omega_n'|\varepsilon}{Q_n}+\frac{\omega_n'^2\varepsilon}{j\omega}+j\omega\varepsilon=\frac{|\omega_n'|\varepsilon}{Q_n}\left[1+jQ_n\left(\frac{\omega}{|\omega_n'|}-\frac{|\omega_n'|}{\omega}\right)\right]=\frac{|\omega_n'|\varepsilon}{Q_n}[1+j\ \tan\ \phi]$$

where we have put

$$\boxed{\tan\ \phi=Q_n\left(\frac{\omega}{|\omega_n'|}-\frac{|\omega_n'|}{\omega}\right)} \tag{6.70}$$

Finally (6.67) is approximated by

$$\frac{j\omega\mu\left(1+\dfrac{1-j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\dfrac{|\omega_n|}{|\omega|}}\right)}{k_n^2-k^2\left(1+\dfrac{1-j\ \mathrm{sgn}(\omega)}{Q_n}\sqrt{\dfrac{|\omega_n|}{|\omega|}}\right)}\approx\frac{1}{1+j\ \tan\ \phi}\cdot\frac{Q_n}{|\omega_n'|\varepsilon}\ . \tag{6.71}$$

Now, by definition,

$$\frac{|\omega_n'|}{Q_n}=\frac{P_n}{W_n}$$

where P_n and W_n represent the power dissipated in the walls, and the energy stored in the cavity for the n^{th} solenoidal mode. From (6.22),

$$W_n=2\ W_{\text{electric}}=\frac{1}{2}\varepsilon\ \int|\vec{E}_n|^2 dV\ .$$

With (6.71) the n^{th} resonant term in (6.65) may thus be written as

$$\boxed{\vec{E}(\omega) = -\frac{1}{1+j\ \tan\ \phi}\frac{\vec{E}_n \int \vec{J}\cdot\vec{E}_n^* dV}{2P_n} + \ldots} \tag{6.72}$$

6.5 Beam coupling impedance $Z_{||}$ and $Z_\perp$

The cavity is excited by the convection current of a charged particle beam; the particles with velocity υ traverse the cavity on the z-axis. The beam current is then a function of (t - z/υ) which is supposed to be modulated at frequency ω; therefore the physical current is

$$\mathrm{Re}\left[I_b(0)\cdot \mathrm{e}^{j\omega\left(t-\frac{z}{\upsilon}\right)}\right] = \mathrm{Re}\left[\mathrm{e}^{j\omega t}\cdot I_b(0)\,\mathrm{e}^{-j\frac{\omega}{\upsilon}z}\right]$$

to which there corresponds a complex amplitude

$$I_b = I_b(0)\mathrm{e}^{-jhz} \qquad \text{where} \qquad h = \frac{\omega}{\upsilon}\ . \tag{6.73}$$

If ω is close to a resonant frequency ω_n' of the cavity, it is sufficient to keep only a single resonant term of the type (6.72).

Longitudinal coupling impedance

If a particle passes the plane $z = 0$ at time t_o, it will pass the plane z at time $t = t_o + z/\upsilon$ (neglecting the change in υ during the traversal of the cavity). The real voltage gain experienced by the particle is

$$\mathrm{Re}\int E_z\ \mathrm{e}^{j\omega\left(t_o+\frac{z}{\upsilon}\right)}dz = \mathrm{Re}\left[\mathrm{e}^{j\omega t}\int E_z\ \mathrm{e}^{jhz}dz\right]\ .$$

Therefore the complex amplitude of the *effective accelerating voltage* reads

$$V_{||} = \int_{-\infty}^{+\infty} E_z\ \mathrm{e}^{jhz}dz\ . \tag{6.74}$$

Since this formula takes into account the finite velocity of the particles, it automatically includes the transit time factor.

Strictly speaking, in (6.74) E_z depends on the transverse coordinates of the particle, which in general vary during the cavity traversal. For the sake of simplicity, in the case of ultrarelativistic particles such a variation is neglected, and the integral (6.74) is taken *at fixed transverse coordinates*.

In fact, the integral (6.74) is the Fourier transform of E_z with respect to z, for a propagation constant $\gamma = jh$. From the Helmholtz equation (3.5) for the Fourier transform of E_z, its transverse variation is given by terms $J_m(k_c r)\cos m\varphi$ similar to (3.87), with k_c defined as in (3.5); that is

$$k_c^2 = k^2 - h^2 = -\frac{\omega^2}{\upsilon^2}\left(1-\frac{\upsilon^2}{c^2}\right) = -\frac{k^2}{\beta^2\gamma^2}$$

where β, γ here represent the relativistic factors for a particle with velocity υ. It follows that

$$V_{||}(r,\varphi) = \sum_{m=0}^{\infty}(2\beta\gamma)^m\cdot I_m\left(\frac{kr}{\beta\gamma}\right)\cdot\left[V_{cm}\cos m\varphi + V_{sm}\sin m\varphi\right] \tag{6.75}$$

where V_{cm}, V_{sm} are independent of r,φ and $I_m(z)$ is the modified Bessel function of the first kind; this Fourier series is valid as long as r does not exceed the minimum radius of the cavity aperture along the z-axis. On the z-axis, the $m = 0$ term is accelerating, whereas the $m = 1$ term is deflecting.

If the cavity has rotational symmetry about the z-axis, the series (6.75) reduces to a single m-term. Then, for an accelerating mode,

$$V_{\|}(r,\varphi) = V_{\|}(r=0) \cdot I_o\left(\frac{kr}{\beta\gamma}\right) . \tag{6.76}$$

In this case of rotational symmetry, using (6.75), for any m $V_{\|}(r,\varphi)$ can be deduced from its value at a single $r > 0$. This is extremely useful in numerical computations, when the cavity has a gap in a beam pipe of inner radius a: it is then sufficient to compute (6.74) at $r = a$, where E_z vanishes everywhere except in the gap.

Using (6.73), it should be noticed that (6.74) can also be written as

$$V_{\|} = \frac{1}{I_b^*(0)} \int E_z I_b^* \; dz \tag{6.77}$$

With (6.73) and (6.74), the driving term due to the beam reads

$$\int \vec{J} \cdot \vec{E}_n^* dV = I_b(0) \int E_{nz}^* \; e^{-jhz} dz = I_b(0) \cdot V_{\|n}^* . \tag{6.78}$$

By definition, the longitudinal impedance seen by the beam is

$$Z_{\|}[\Omega] = -\frac{V_{\|b}}{I_b(0)} \tag{6.79}$$

where $V_{\|b}$ is the effective accelerating voltage induced by the beam itself.

Using (6.72), (6.74) and (6.78) it becomes

$$Z_{\|} = \frac{1}{1 + j \, \tan \, \phi} \; \frac{\left| \int E_{nz} \; e^{jhz} dz \right|^2}{2P_n}$$

Remembering (6.70) this may be rewritten as

$$\boxed{\begin{aligned} & Z_{\|}[\Omega] = \frac{R_{\|n}}{1 + jQ_n \left(\dfrac{\omega}{|\omega_n'|} - \dfrac{|\omega_n'|}{\omega} \right)} \\ & \text{where} \\ & R_{\|n}[\Omega] = \frac{\left| \int E_{nz} \; e^{jhz} dz \right|^2}{2P_n} = \frac{|V_{\|n}|^2}{2P_n} \end{aligned}} \tag{6.80}$$

$R_{\|}$ is the *longitudinal shunt resistance* of the cavity at its n^{th} resonant frequency; it is usually taken on the cavity axis. For the beam, in the vicinity of a resonance, the cavity behaves as a parallel resonant circuit.

Remarks:

1) In (6.80), the shunt resistance $R_{\|}$ is defined in terms of the power loss P in the cavity walls for a given level $|V_{\|}|$ of the effective accelerating voltage on the cavity axis. This definition holds irrespective of

RF source; in (6.80) the RF source is supposed to be the beam, but the same definition applies when the RF source is an external generator.

2) The shunt resistance $R_{\|}$ is defined in (6.80) by the same formula

$$R_{\|} = \frac{|V_{\|}|^2}{2P}$$

which is used in the theory of electric circuits. Although this definition was also used for the first proton linac [37], the builders of the early electron linacs in Stanford [38] defined the shunt resistance as

$$\text{"linac"}\ R_{\|} = \frac{|V_{\|}|^2}{P} .$$

Since they have been widely followed in the Western world (but not in the USSR), one should never forget that

$$\boxed{(\text{linac } R_{\|}) \ = \ 2\ (\text{circuit } R_{\|})}$$

Transverse coupling impedance

In (3.14), if we use $\beta = \upsilon/c$ instead of υ in the integral giving the transverse momentum gained by a unit charge when traversing a cavity, this integral acquires the dimensions of a voltage, which is called the *effective deflecting voltage* produced by the cavity:

$$\vec{V}_{\perp} = \int \vec{F}_{\perp}\ e^{jhz} \frac{dz}{\beta}$$

The integral is supposed to be taken in the beam pipe, from a region upstream to a region downstream of the cavity, at distances far enough from the cavity for the electromagnetic fields to be negligible. Then from the Panofsky-Wenzel theorem (3.14) we have

$$\vec{V}_{\perp} = -\frac{1}{jk} \int \mathrm{grad}_{\perp} E_z \cdot e^{jhz} dz \ . \tag{6.81}$$

In general $\mathrm{grad}_{\perp} E_z$ depends on the transverse coordinates of the particle, which vary during the cavity traversal. As in (6.74), such a variation is again neglected, and the integral (6.81) is taken *at fixed transverse coordinates*. In this case (6.81) and (6.74) are related by:

$$\vec{V}_{\perp} = -\frac{1}{jk} \mathrm{grad}_{\perp}(V_{\|}) \ . \tag{6.82}$$

For a mode deflecting in the x-direction, E_z is zero on the z-axis and reverses sign with x; moreover it is approximately independent of y. Its Taylor expansion reads

$$E_z = x \cdot \left.\frac{\partial E_z}{\partial x}\right|_o + \text{higher order terms} \ . \tag{6.83}$$

For a beam close to the z-axis, (6.81) then becomes

$$V_x = -\frac{1}{jk} \int \left.\frac{\partial E_z}{\partial x}\right|_o e^{jhz} dz \qquad \text{taken on } z\text{ - axis} \ . \tag{6.84}$$

If the beam traverses the cavity at a fixed distance x_o off-axis, with (6.73) and (6.83) the driving term reads

$$\int \vec{J} \cdot \vec{E}_n^* dV = I_b(0) x_o \int \left. \frac{\partial E_{nz}^*}{\partial x} \right|_o e^{-jhz} dz \; . \tag{6.85}$$

By definition, the transverse coupling impedance seen by the beam is

$$Z_x\left[\Omega \mathrm{m}^{-1}\right] = j \frac{V_x}{I_b(0) x_o} \tag{6.86}$$

where $I_b(0)x_o$ is the dipole moment of the beam current.

Using (6.72), (6.84) and (6.85) it becomes

$$Z_x = \frac{1}{k} \frac{1}{1+j \tan \phi} \frac{\left| \int \left. \frac{\partial E_{nz}}{\partial x} \right|_o e^{jhz} dz \right|^2}{2P_n} = \frac{k_n'^2}{k} \frac{1}{1+j \tan \phi} \frac{|V_{xn}|^2}{2P_n}$$

Remembering (6.70) this may be rewritten as

$$\boxed{\begin{gathered} Z_\perp\left[\Omega \mathrm{m}^{-1}\right] = \frac{k_n'^2}{k} \frac{R_{\perp n}}{1 + jQ_n \left(\frac{\omega}{|\omega_n'|} - \frac{|\omega_n'|}{\omega} \right)} \\ \text{where} \\ R_{\perp n}[\Omega] = \frac{\left| \frac{1}{k_n'} \int \left. grad_\perp E_{nz} \right|_o e^{jhz} dz \right|^2}{2P_n} = \frac{|V_{\perp n}|^2}{2P_n} \end{gathered}} \tag{6.87}$$

$R_\perp$ is the *transverse shunt resistance* of the cavity at its n^{th} resonant frequency; it is taken on the cavity axis.

At resonance the transverse impedance (6.87) is real and positive; it is in order to achieve this result that the factor j appears in the definition (6.86).

Moreover, at resonance (6.87) yields the important relation

$$\boxed{Z_\perp\left[\Omega \mathrm{m}^{-1}\right] = k_n'\left[\mathrm{m}^{-1}\right] \cdot R_\perp[\Omega] \quad \text{where} \quad k_n' = \frac{\omega_n'}{c}} \tag{6.88}$$

Some authors use (6.81) without k in the denominator of $\vec{V}_\perp$; then, in (6.87) one uses

$$R_{\perp n}\left[\Omega \mathrm{m}^{-2}\right] = k_n'^2\left[\mathrm{m}^{-2}\right] \cdot R_{\perp n}[\Omega] \; .$$

Remarks

1) In (6.87) the shunt resistance $R_\perp$ is defined in terms of the power loss P in the cavity walls for a given level $|V_\perp|$ of the deflecting voltage on the cavity axis. Again this definition holds irrespective of the RF source; in (6.87) the RF source is supposed to be the beam, but the same definition applies when the RF source is an external generator. In fact, the latter definition has been used as a figure of merit for RF separators of high energy particles ([19], p. 268). The same shunt resistance $R_\perp$ appears in the theory of beam breakup in electron linacs.

2) As for the longitudinal shunt resistance, there is a "linac" definition of $R_\perp$ as

$$\text{"linac" } R_\perp = \frac{|V_\perp|^2}{P}$$

instead of the "circuit" definition used in (6.87). Therefore one should not forget that

$$\boxed{(\text{linac } R_\perp) = 2(\text{circuit } R_\perp)}$$

Finally, it is worthwhile noticing that when used in a deflecting mode, a cavity constitutes an excellent transverse pick-up for the beam; in that case $R_\perp$ should not be too large in order to prevent beam instabilities.

Example: Transverse impedance due to a gap in a circular vacuum chamber

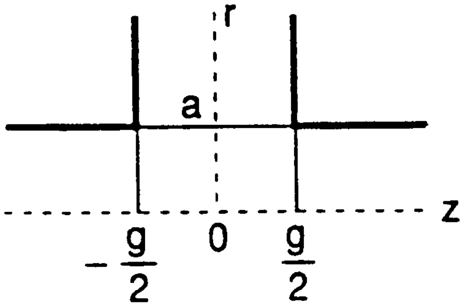


Fig. 6.2 A gap g in a circular chamber of radius a

An off-axis beam at polar coordinates (r_o, φ_o) excites all modes of (6.75). We now consider only the dipole mode $m = 1$:

$$V_\parallel(r,\varphi) = V_\parallel(a)\frac{I_1(\kappa r)}{I_1(\kappa a)}\cos(\varphi - \varphi_o) \quad \text{where} \quad \kappa = \frac{k}{\beta\gamma} \tag{6.89}$$

and

$$V_\parallel(a) = V_{\text{gap}}\frac{\sin\left(\frac{\omega}{\upsilon}\frac{g}{2}\right)}{\left(\frac{\omega}{\upsilon}\frac{g}{2}\right)}$$

is the amplitude (with respect to φ) of the effective voltage induced by the beam at $r = a$; the gap induced voltage V_{gap} can in principle be computed once the surroundings at $r > a$ are known.

The transverse deflecting voltage is then obtained by applying (6.82) to (6.89):

$$\begin{aligned} V_r &= -\frac{1}{jka}\, V_\parallel(a)\frac{\kappa a}{I_1(\kappa a)}I_1'(\kappa r)\, \cos(\varphi - \varphi_o) = -\frac{V_\parallel(a)}{jka}\,\frac{I_o(\kappa r) + I_2(\kappa r)}{I_o(\kappa a) - I_2(\kappa a)}\, \cos(\varphi - \varphi_o) \\ V_\varphi &= -\frac{1}{jkr}\,\frac{\partial}{\partial\varphi}V_\parallel(r) = \frac{V_\parallel(a)}{jka}\,\frac{\kappa a}{\kappa r}\,\frac{I_1(\kappa r)}{I_1(\kappa a)}\, \sin(\varphi - \varphi_o) \end{aligned} \tag{6.90}$$

For ultrarelativistic particles, $\beta\gamma >> 1$ and $\kappa a = ka/(\beta\gamma) << 1$; (6.90) then reduces to

$$V_r = -\frac{V_\parallel(a)}{jka}\cos(\varphi - \varphi_o) \qquad V_\varphi = \frac{V_\parallel(a)}{jka}\sin(\varphi - \varphi_o)$$

hence

$$\vec{V}_\perp = V_r \vec{1}_r + V_\varphi \vec{1}_\varphi = -\frac{V_\parallel(a)}{jka}\, \vec{1}_{ro} \tag{6.91}$$

From (6.86) we obtain

$$\vec{Z}_\perp\left[\Omega\mathrm{m}^{-1}\right] = \left[-\frac{1}{ka}\,\frac{V_\parallel(a)}{I_b(0)r_o}\right] \vec{1}_{ro} \tag{6.92}$$

where the ratio $V_\parallel(a)/[I_b(0)r_o]$ is independent of the beam dipole moment $I_b(0)\vec{r}_o$. It appears that the transverse kick experienced by a unit charge is in the direction of the beam offset $\vec{r}_o$.

Multipolar modes

The electric field (6.83) corresponds to $m = 1$ in (6.75). For $m > 1$, from (6.74) and (6.75) the effective voltage $V_{\parallel n}$ in the n^{th} mode has the form

$$V_{\parallel n} = \int E_{nz}\; e^{jhz} dz = K_n\; r^m \,{}^{\cos}_{\sin}(m\varphi) \;+\; \text{higher order terms} \tag{6.93}$$

where $\cos m\varphi$, $\sin m\varphi$ correspond to the two different polarizations of the same mode. For a beam passing at (r_o, φ_o) the driving term reads, with (6.78):

$$\int \vec{J}\cdot\vec{E}_n^* dV = I_b(0)\cdot\left[V_{\parallel n}^* \text{ at the beam position}\right] = I_b(0)\cdot K_n^*\; r_o^m \,{}^{\cos}_{\sin}(m\varphi_0) \;+\; \text{higher order terms} \tag{6.94}$$

Summing up the contributions of the two mode polarizations we obtain

$$\begin{aligned} V_{\parallel n}\int \vec{J}\cdot\vec{E}_n^* dV &= I_b(0)\;|K_n|^2\; r^m r_o^m\; \left[\cos(m\varphi)\cos(m\varphi_0) + \sin(m\varphi)\sin(m\varphi_0)\right] \\ &= I_b(0)\;|K_n|^2\; r^m r_o^m \cos m(\varphi - \varphi_0) \end{aligned} \tag{6.95}$$

Finally, with (6.72) the effective accelerating voltage induced by the beam reads

$$V_{\parallel b} = -\;\frac{1}{1 + j\;\tan\;\phi}\;\frac{|K_n|^2}{2P_n}\; I_b(0)\; r^m r_o^m \cos m(\varphi - \varphi_0) \tag{6.96}$$

By analogy wtih (6.79), the *longitudinal coupling impedance* is defined as

$$Z_\parallel\left[\Omega\;\mathrm{L}^{-2m}\right] = -\frac{V_{\parallel b}}{I_b(0)\; r^m r_o^m \cos m(\varphi - \varphi_0)} \tag{6.97}$$

where L represents the unit of length (to avoid confusion with the multipole number m). Therefore, from (6.96):

$$\boxed{Z_\parallel\left[\Omega\mathrm{L}^{-2m}\right] = \frac{1}{1 + j\;\tan\;\phi}\;\frac{|K_n|^2}{2P_n}} \qquad m = 0,\,1,\,2,\,... \tag{6.98}$$

where K_n is defined by (6.93). This formula generalizes (6.80) to any m.

A longitudinal impedance expressed in Ω can be defined as

$$Z_\parallel[\Omega] = \frac{Z_\parallel\left[\Omega\mathrm{L}^{-2m}\right]}{k_n^{2m}} \tag{6.99}$$

The transverse deflecting voltage is obtained by applying (6.82) to (6.96):

$$\vec{V}_{\perp b} = \frac{1}{jk}\,\frac{1}{1+j\,\tan\,\phi}\,\frac{|K_n|^2}{2P_n} I_b(0)\; m r^{m-1} r_o^m \quad \underbrace{\left[\vec{1}_r \cos m(\varphi-\varphi_0) - \vec{1}_\varphi \sin m(\varphi-\varphi_0)\right]}_{\left[\vec{1}_{r0} \cos\,(m-1)(\varphi-\varphi_0) - \vec{1}_{\varphi o} \sin\,(m-1)(\varphi-\varphi_0)\right]} \tag{6.100}$$

The vector inside the square bracket is simple only when $m = 0$ (it then reduces to $\vec{1}_r$)and when $m = 1$ (it then reduces to $\vec{1}_{ro}$.

By analogy with (6.86), the *transverse coupling impedance* is defined as

$$\vec{Z}_\perp\left[\Omega\mathrm{L}^{-2m+1}\right] = j\frac{V_{\perp b}}{I_b(0)\; m r^{m-1} r_o^m} \qquad m = 1,2,... \tag{6.101}$$

With (6.100) and (6.98) it reads

$$\vec{Z}_\perp\left[\Omega\mathrm{L}^{-2m+1}\right] = \frac{1}{k} Z_\parallel\left[\Omega\mathrm{L}^{-2m}\right]\cdot\left[\vec{1}_r \cos m(\varphi-\varphi_0) - \vec{1}_\varphi \sin m(\varphi-\varphi_0)\right] \quad \text{for a given } m \tag{6.102}$$

A transverse impedance expressed in Ω can be defined as

$$\vec{Z}_\perp[\Omega] = \frac{\vec{Z}_\perp\left[\Omega\mathrm{L}^{-2m+1}\right]}{k_n^{2m-1}} \tag{6.103}$$

With (6.99) the relation (6.102) then becomes

$$\vec{Z}_\perp[\Omega] = \frac{k_n}{k} Z_\parallel[\Omega]\cdot\left[\vec{1}_r \cos m(\varphi-\varphi_0) - \vec{1}_\varphi \sin m(\varphi-\varphi_0)\right] \tag{6.104}$$

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