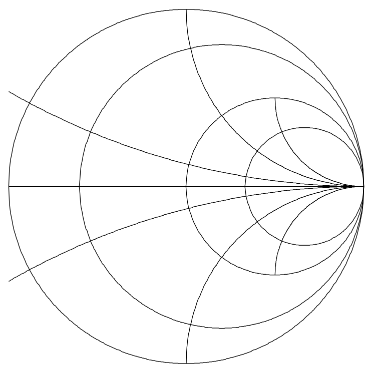

Microwave Measurements Laboratory

USPAS and CCAST
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RF Cavity higher order mode measurements

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(adapted from notes by R. Rimmer)

•Introduces the student to the radio frequency (RF) cavity, by means of equivalent circuit models and a simple pillbox example. Introduces resonant modes, shunt impedance, quality factor, transit-time factor, coupling factor, loaded and external Q, transverse impedance.

•Introduces the student to cavity measurements in transmission and relfection and field mapping by the perturbation method

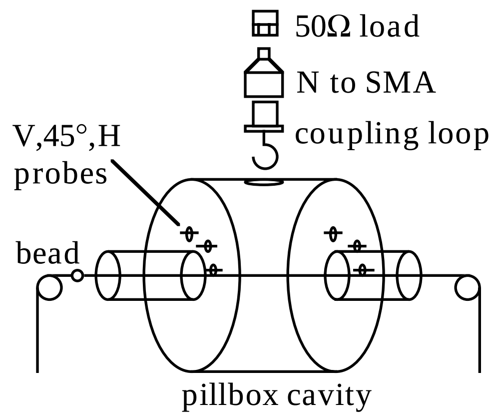
•Introduces the student to mode measurements, quality factor, coupling factor, field profile, shunt impedance.

Introduction

The experiment is designed to introduce the student to the concept of cavity modes by studying the mode spectrum of a simple model cavity using a network analyzer (NWA). The first few modes are identified by frequency and their Q's measured by a transmission method. For the TM_{010} mode, which would be the accelerating mode in a real cavity, the coupling through a drive loop is determined by means of a reflection measurement and the loaded and unloaded Q's are calculated. The loading of the higher-order modes (HOMs) is also observed and their Q reduction is measured. Brief background material is also included.

Equipment:

Aluminum model pillbox cavity
Network Analyzer + calibration kit
2 cables
2 N-type to SMA adapters
2 50Ω SMA loads
Coupling loop
E-Field probe
1 SMA F-F connector (for Thru calibration)
Bead-pull apparatus



Experiment:

Part 1: Transmission measurement

•Connect the network analyzer to the probes on each end of the cavity to make a transmission measurement (without coupling loop.) Observe Log $|S_{21}|$ on the NWA, set the frequency range from 700 MHz to 2.3 GHz and adjust the scale so that the resonant peaks in the spectrum are clearly visible. Plot the transmission response between the vertical pair of probes on the printer or plotter and save in the NWA memory. Connect the cables to the horizontal pair and compare the data with the memory. Are there any significant differences?

•Record the frequencies of the peaks and compare with the mode frequencies in table 1, which are calculated by a cavity design program (URMEL).

•Make a table in order of measured frequency identifying each mode with it's URMEL counterpart and with additional columns for measurements of unloaded and loaded Q's. Zoom close in on a peak corresponding to a dipole mode and again compare vertical and horizontal spectra.

•Record the unloaded Q (Qo) for each mode. (Do only for vertical probe pair)

Table 1: Mode frequencies calculated by URMEL

mode type	Freq (MHz)	R/Q* (Ω)
0E1	786	100.757
1E1	1245	9.761
1M1	1321	0.107
0M1	1417	14.298
2M1	1549	0.000
2E1	1675	0.227
1M2	1705	8.345
0E2	1808	8.004
2M2	2048	0.570
1M3	2104	0.092
0M2	2162	28.562
1E2	2272	0.045

(mode type = aE/Mb where a is the azimuthal order (0=monopole, 1=dipole, etc.) E/M indicates the longitudinal mode symmetry (E=electric boundary condition in midplane, M=magnetic), b= URMEL solution number. *R/Q is calculated on axis for monopole modes and at the beam-pipe radius for higher order modes)

Part2: Coupling measurement

•Insert the coupling loop, remove the cables from the transmission probes and connect port 1 of the NWA to the coupler. Observe S_{11} for the TM_{010} (0E1) mode through the range of probe angles from zero to maximum coupling and choose a frequency span that is large enough to accomodate the shift in frequency but narrow enough to allow good resolution of the resonance curve. Calibrate port 1 of the NWA over this frequency range. Observe the response with calibration turned on for S_{11} or VSWR and on the smith chart as the coupler is rotated. (Note that when using the smith chart the electrical delay setting on the network analyzer should be adjusted so that the resonant frequency lies on the real axis and the appropriate reactance is obtained above and below resonance).

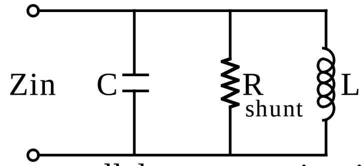
•Measure and plot β at resonance as a function of coupler angle. (What variation would one expect from a simple consideration of the loop area coupled to the azimuthal magnetic field?). Determine β_{max} and the angle for the best match (β=1).

•Set the coupler in the matched position and lock in place. Measure β accurately from the $|S_{11}|$ or VSWR curve, calculate the appropriate values for the half-power points for Qo and QL and measure the bandwidth at these values. Record the Qo and Ql. Check if Qo/QL=(1+β). Does Qo measured this way agree with the transmission measurement? Measure Qo again from the half-power points on the smith chart. How does this compare?

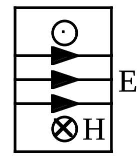
Part 3: Loaded Q measurement

Disconnect the cable from the coupler and replace with a 50Ω load. Go back to the transmission measurement set-up and re-plot the spectra for the three probe orientations. How are they different from before? Measure the loaded Q's in the Vertical and Horizontal orientations and add them to the table.

Properties of parallel resonant circuit and real cavity:



parallel resonant circuit



cavity

The input impedance of the equivalent circuit can be expressed as:

$$Z_{in} = \left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right)^{-1}$$

$$= \frac{R}{1+jQ\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)} \approx \frac{R}{1+jQ2\left(\frac{\delta\omega}{\omega_o}\right)}$$

where $\omega_o = \frac{1}{\sqrt{LC}}$ and $Q=\omega_o RC$ or $Q=\frac{R}{\omega_o L}$ (see homework problem #1).

For a cavity the **shunt impedance** , **R**, is defined in terms of the voltage produced in the cavity for a given power dissipation, $\mathbf{R=V^2/2P}$, where the voltage is considered as the integral of the electric field along the flight path of a particle, $\mathbf{V=\int E.dl}$. N.b.: In some physics texts the shunt impedance is defined as $R=V^2/P$ and may or may not include the transit-time factor.

The **quality factor Q** of the cavity or equivalent circuit is a measure of the sharpness of the resonance and also of the enhancement of the voltage and current compared to a simple traveling wave. The Q is defined as the ratio between the stored energy and the power dissipation per radian or $\mathbf{Q=\omega U/P}$

The ratio **R/Q** is a figure of merit for the shape of the cavity and is independent of the material and it can be shown also that

$$\frac{R}{Q} = \sqrt{\frac{L}{C}} = \frac{1}{\omega C} = \omega L$$

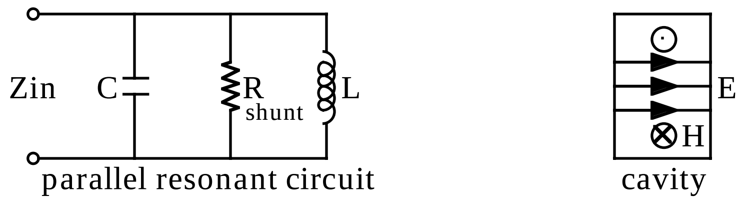
which gives a clue as to how to optimize the cavity shape once the field distributions are known (see section on “real cavities”).

If the time taken for a particle to cross the cavity is a significant fraction of the RF period then the effective voltage seen or induced by the particle is reduced because of the $\cos(\omega t)$ time dependence of the fields. The factor by which it is reduced is called the **transit-time factor**, **T**, and is defined as the ratio of the energy actually received to that which would be received if the field were constant at the maximum value. The **longitudinal beam impedance** $\mathbf{Z_{||}}$ (sometimes $R_{||}$) is the product of the shunt impedance and the square of the transit-time factor,

$$Z_{||} = RT^2$$

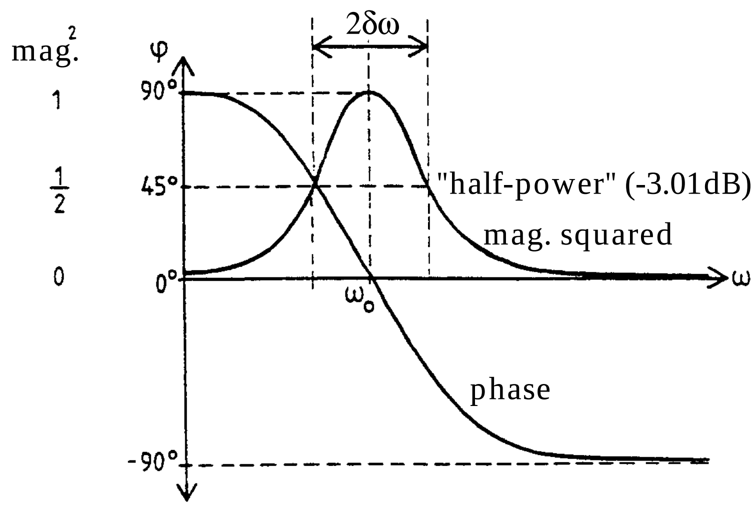
(note: this is different from the impedance of the cavity as a pick-up which is $RT^2/4$).

Magnitude and phase of parallel circuit and cavity:



The resonant nature of the circuit is clearly seen by looking at the magnitude and phase of the impedance as a function of frequency.

Note that at resonance the impedance is purely resistive and equal to the shunt impedance. Below resonance the impedance is inductive and above resonance it is capacitive. In the cavity there will be other modes with higher resonant frequencies and the impedance will become inductive, resistive and capacitive again for each in turn as the frequency increases.



Transmission (S_{21}) measurement:

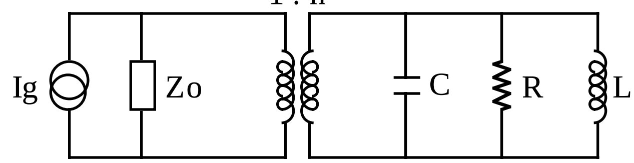
The impedance response can be measured by the transmission between two probes coupled to the cavity. If the probes are weakly coupled then the Q is not changed significantly, otherwise the coupling factor of the probes must be measured and taken into account.

The Q can be measured from the resonance curve by taking the bandwidth at the half-power (1/√2 voltage, or -3.01dB) points.

$$Q = \frac{\omega_o}{2\delta\omega}$$

Coupling to matched source/load:

To supply energy to the cavity from an external source or extract signal power induced by the beam requires a means of coupling the cavity fields to an external circuit. This can be represented in the equivalent circuit by an idealized transformer of turns ratio 1:n linking the cavity voltage to a transmission line which is matched to an ideal current source representing the generator.



This circuit can be used to transform the cavity impedance into the transmission line to observe the load presented to the generator (fig a) or to transform the generator current and source impedance into the cavity so that the total cavity voltage from generator- and beam-induced currents can be calculated (fig. b).

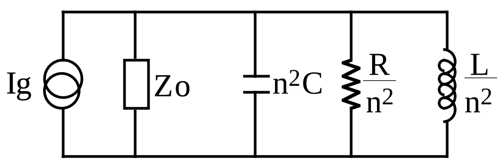


fig. a cavity referred to input

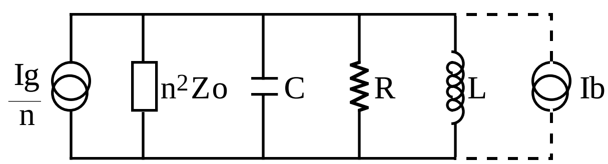


fig. b source referred to cavity

The **coupling factor** β is defined as the ratio of the power loss in the external circuit to that in the cavity. A loaded Q , $\mathbf{Q_L}$, can be defined as the ratio of the stored energy to the total loss per radian, and an external Q, $\mathbf{Q_{ext}}$, can be defined as the ratio of the stored energy to the loss in the external circuit per radian. It can thus be shown that:

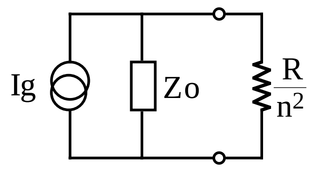
$$\beta = \frac{\text{power loss in ext. cct}}{\text{power loss in cavity}} = \frac{Q_o}{Q_{ext}} = \frac{R}{n^2 Z_o}\ ,\ \frac{1}{Q_L} = \frac{1}{Q_o} + \frac{1}{Q_{ext}}\ , \text{and } Q_o = (1+\beta)Q_L$$

When the cavity is matched to the source at resonance (without beam):

$$\beta = 1,\ Q_L = \frac{Q_o}{2}, \text{ and } n^2 = \frac{R}{Z_o}$$

Reflection coefficient looking toward cavity at resonance:

Away from resonance most of the power incident on the cavity is reflected but close to resonance the response may come closer to or go through a matched condition, depending on the coupling. At resonance the resistive impedance R is transformed in to the external circuit so the reflection coefficent is simple to calculate:



cavity impedance at resonance referred to input

Reflection coefficent $$\Gamma = \frac{\frac{R}{n^2} - Z_o}{\frac{R}{n^2} + Z_o} = \frac{\beta - 1}{\beta + 1}$$

while $$VSWR = \frac{1+|\Gamma|}{1-|\Gamma|}$$

so $0<\beta<1$ (under-coupled) VSWR = $1/\beta$

$\beta=1$ (matched) VSWR = 1

$\beta>1$ (over-coupled) VSWR = β

Measurement of cavity properties from S_{11}:

Provided the losses in the coupler can be neglected*, the coupling factor β, and the loaded and unloaded Q's can be calculated from an accurate measurement of S_{11} looking towards the cavity. As described above, the coupling factor can be determined from the reflection coefficient at resonance. If the coupling can be adjusted it is often a simple matter to determine whether the system is under- or over- coupled. If the VSWR dips towards unity through the range of adjustment but a match is never achieved then the system is undercoupled (and $\beta_{max} = 1/VSWR_{min}$). If the VSWR goes to 1 then rises to a local maximum the system is overcoupled (and β_{max} = local max. VSWR). With the phase information available on the network analyzer the complex impedance can be plotted on a Smith chart and it is easily determined whether the system is under- or over-coupled. Once the coupling factor is known the value of S_{11} or the VSWR at the loaded or unloaded half-

* the coupler loss and self-inductance are not represented in this simple equivalent circuit

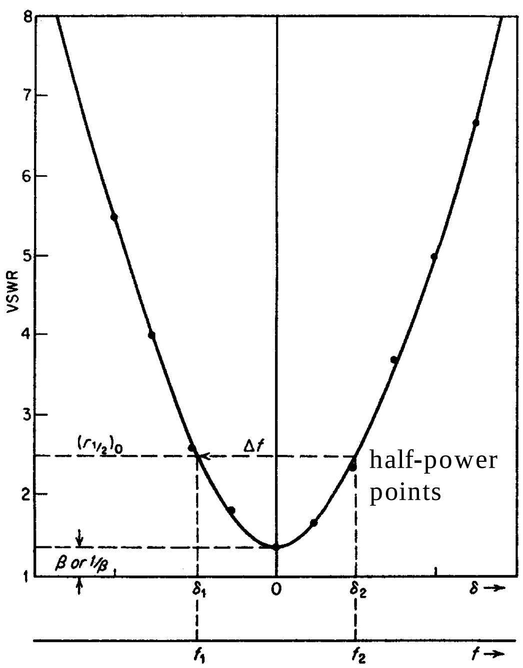
power points can be calculated, or these points can be found on the Smith chart (once the electrical delay has been adjusted to refer the impedance to the detuned-short position)

For the unloaded half-power points (Q_o):

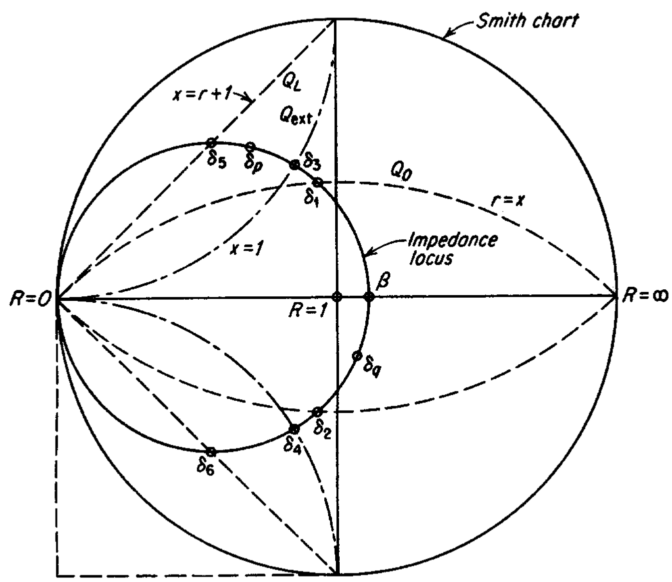
$$|S_{11}| = \sqrt{\frac{5S_o^2-2S_o+1}{S_o^2-2S_o+5}} \qquad \mathrm{VSWR} = \frac{2+\beta^2+\sqrt{4+\beta^4}}{2\beta}$$

For the loaded half-power points (Q_L):

$$|S_{11}| = \sqrt{\frac{S_o+1}{2}} \qquad \mathrm{VSWR} = \frac{1+\beta+\beta^2+(1+\beta)\sqrt{1+\beta^2}}{\beta}$$



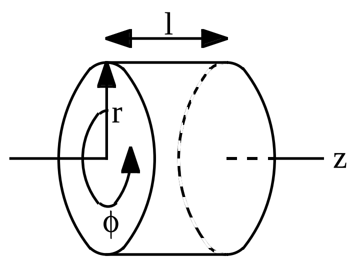
VSWR close to resonance



Identification of the half-power points from the Smith chart. Q_o locus is given by X=R; Q_L by X=R+1, Q_{ext} by X=1

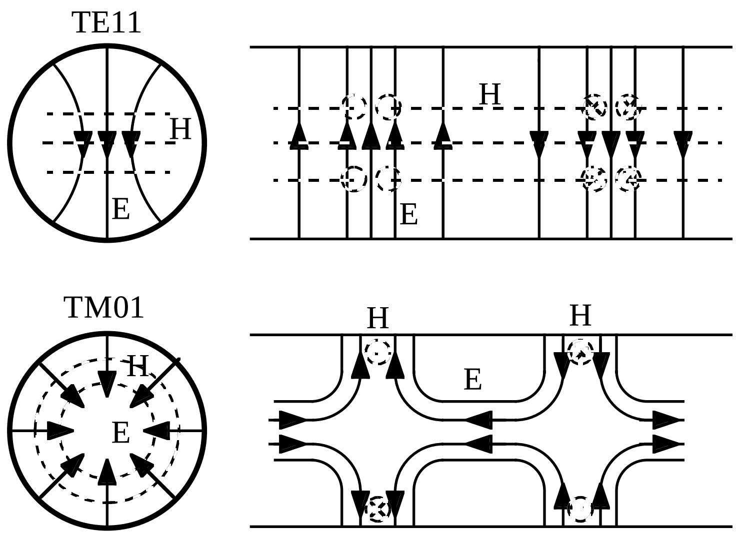
"Pillbox" cavity modes:

The “pillbox” is a simple closed shape for which analytical solutions can be derived for the field and current distributions of the resonant modes. Such a shape could in fact be used as an accelerating structure, however more efficient shapes are usually used in practice. Study of the modes of the pillbox is instructive however and provides much of the nomenclature that is used to describe modes in other axis-symetric structures.



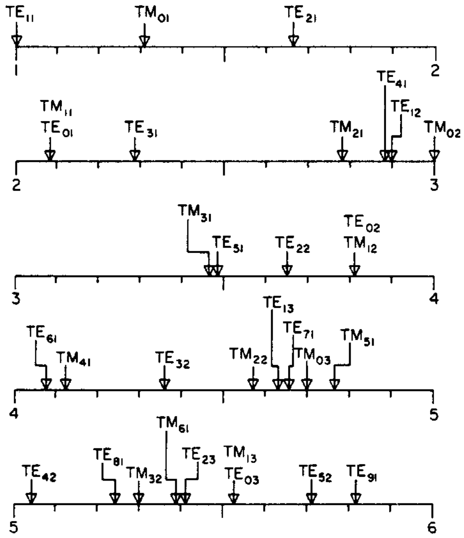
pillbox cavity

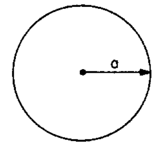
As presented above in the transmission-line analogy, cavity modes can be thought of as resonances between two short circuit planes in a waveguide. In the case of the pillbox this is a length of circular waveguide guide with a short-circuit boundary condition at each end, so the solutions are standing-waves of the TE and TM circular waveguide modes with an integer number of half-wavelengths between the end-plates. The boundary conditions also allow for TM modes with zero variation in the z axis, which are of particular interest for accelerator cavities. The waveguide modes (TE/M_{mn}) are denoted by two subscripts, the first is the number of full periods in ϕ and the second is the number of radial zeros in the field. For cavity modes a third subscript is added which is the number of half-period variations in the z direction.



First two modes in circular waveguide

The following chart shows the cut-off frequencies for modes in circular waveguide, normalized to that of the lowest mode (TE11).



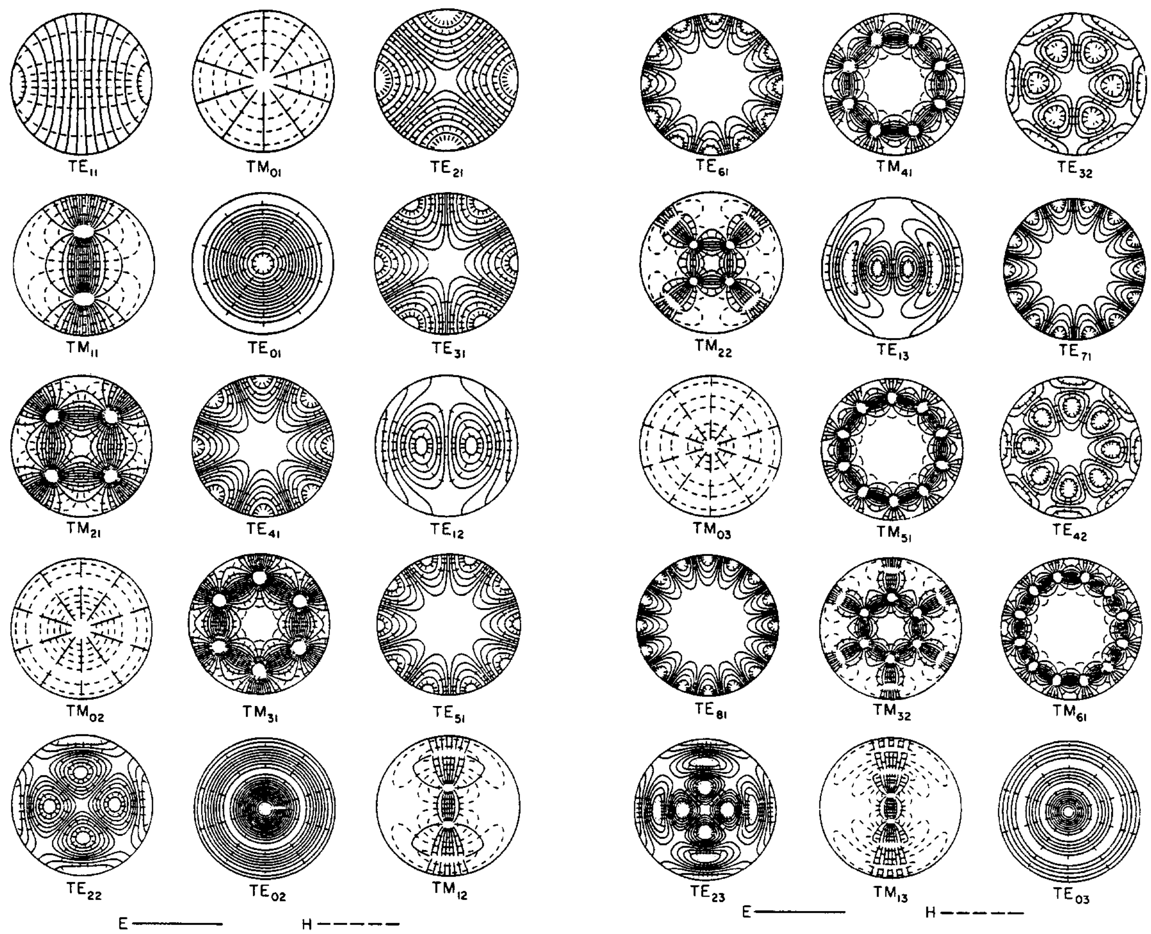


$$kc_{TE11} = \frac{1.841}{a}$$

$$\lambda c_{TE11} = 3.41a$$

$$fc_{TE11} = \frac{0.293}{a\sqrt{\mu\varepsilon}}$$

The figures below show plots of the E and H fields for the first thirty modes [Lee et.al., IEEE Trans. MTT, vol. MTT-33, No. 3, March 1985, p 274].



Only those modes with a component of electric field in the direction of motion of the particle can interact with the beam (Panofsky-Wenzel). For the pillbox this means only the TM modes are of interest. The transverse variations of the longitudinal field are solutions of Maxwell's equations within a circular boundary condition and are Bessel functions of the first kind.

$$E_z(r,\phi) = E_o J_m(k_{mn} r) \cos m\phi$$

where: J_m are the first order Bessel functions
$k_{mn} = x_{mn}/r$ is the transverse wave number
x_{mn} are the roots of the Bessel functions J_m

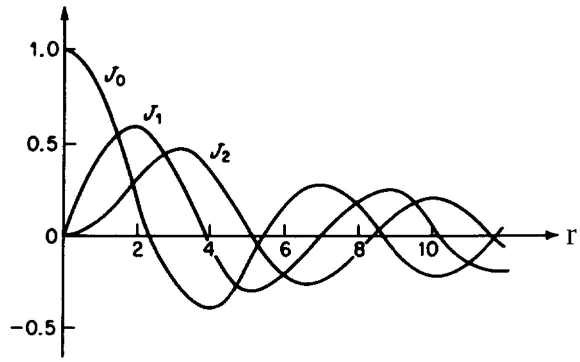
For modes with $E_z(z)$ = constant ($k_z = 0$), $\omega = ck_{mn}$

For modes with $E_z(z) \propto \cos(k_z.z)$,where k_z is the axial wave number:

$$k_o^2 = k_{mn}^2 + k_z^2 \text{ or } \omega_o = c\sqrt{k_{mn}^2 + k_z^2}$$

For TM_{mnz} modes the fields are thus:

$$E_z(r,z,t,\phi) = E_o J_m\left(\frac{x_{mn}}{a}r\right)e^{j\omega t}\cos(m\phi)\cos(k_z z)$$
$$H_\phi(r,z,t,\phi) = H_o J'_m\left(\frac{x_{mn}}{a}r\right)e^{j\omega t}\cos(m\phi)\cos(k_z z)$$



Low-order Bessel functions of the first kind

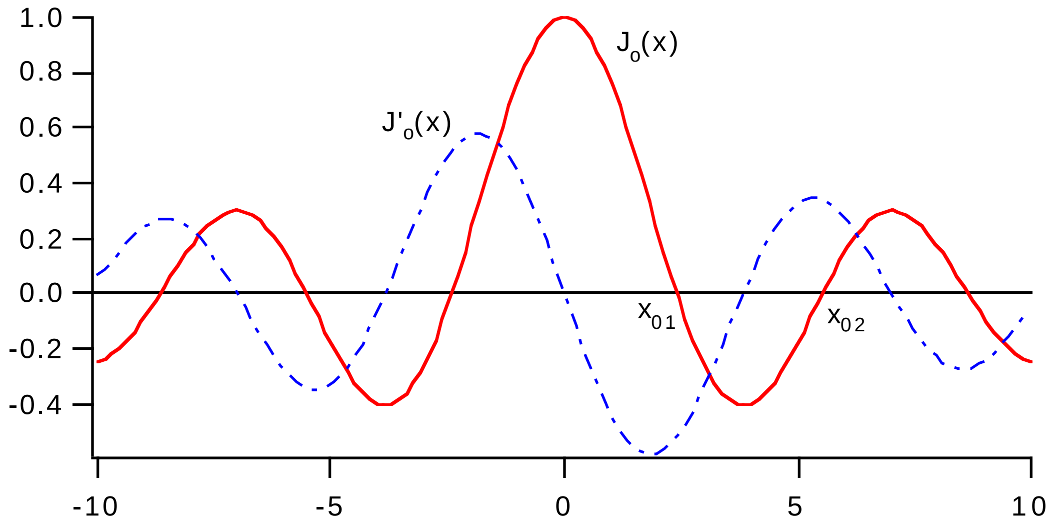
ZEROS AND ASSOCIATED VALUES OF BESSEL FUNCTIONS AND THEIR DERIVATIVES

s	$j_{0,s}$	$J'_0(j_{0,s})$	$j_{1,s}$	$J'_1(j_{1,s})$	$j_{2,s}$	$J'_2(j_{2,s})$
1	2.40482 55577	−0.51914 74973	3.83171	−0.40276	5.13562	−0.33967
2	5.52007 81103	+0.34026 48065	7.01559	+0.30012	8.41724	+0.27138
3	8.65372 79129	−0.27145 22999	10.17347	−0.24970	11.61984	−0.23244
4	11.79153 44391	+0.23245 98314	13.32369	+0.21836	14.79595	+0.20654
5	14.93091 77086	−0.20654 64331	16.47063	−0.19647	17.95982	−0.18773

s	$j_{3,s}$	$J'_3(j_{3,s})$	$j_{4,s}$	$J'_4(j_{4,s})$	$j_{5,s}$	$J'_5(j_{5,s})$
1	6.38016	−0.29827	7.58834	−0.26836	8.77148	−0.24543
2	9.76102	+0.24942	11.06471	+0.23188	12.33860	+0.21743
3	13.01520	−0.21828	14.37254	−0.20636	15.70017	−0.19615
4	16.22347	+0.19644	17.61597	+0.18766	18.98013	+0.17993
5	19.40942	−0.18005	20.82693	−0.17323	22.21780	−0.16712

Monopole modes (m=0):

Modes which have no azimuthal variation are labelled “monopole” modes and TM modes of this type have longitudinal electric field on axis and thus can interact strongly with the beam. The radial distribution of E_z follows J_o, where the zeros satisfy the boundary condition that $E_z = 0$ at the conducting wall at radius a. Similarly H_ϕ and E_r (if present) follow J'_o and are zero in the center and have a finite value at the wall.



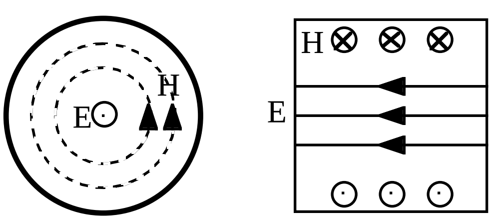
For TM_{oni} modes:

$E_z = E_o J_o(k_{on} r)\cos(k_z z)$ where $k_{on} = x_{on}/a$ and $k_z = i\pi/\text{length}$ $(i \geq 0)$

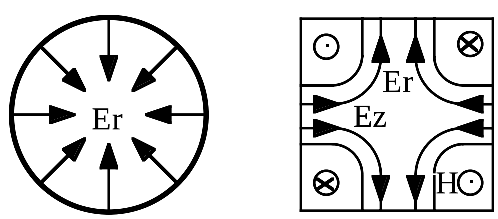
$H_\phi = H_{\phi o} J'_o(k_{on} r)\cos(k_z z)$ $\qquad x_{o1} = 2.405$

$E_r = E_{ro} J'_o(k_{on} r)\sin(k_z z)$ $\qquad x_{o2} = 5.520$

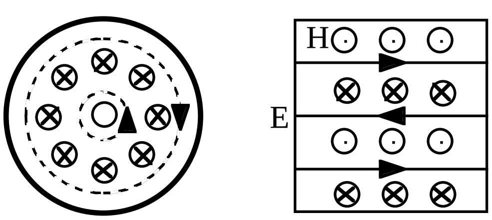
$x_{o3} = 8.654$



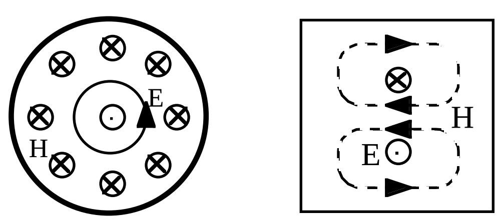
TM_{010}



TM_{011}



TM_{020}



TE_{011}

Dipole modes (m=1):

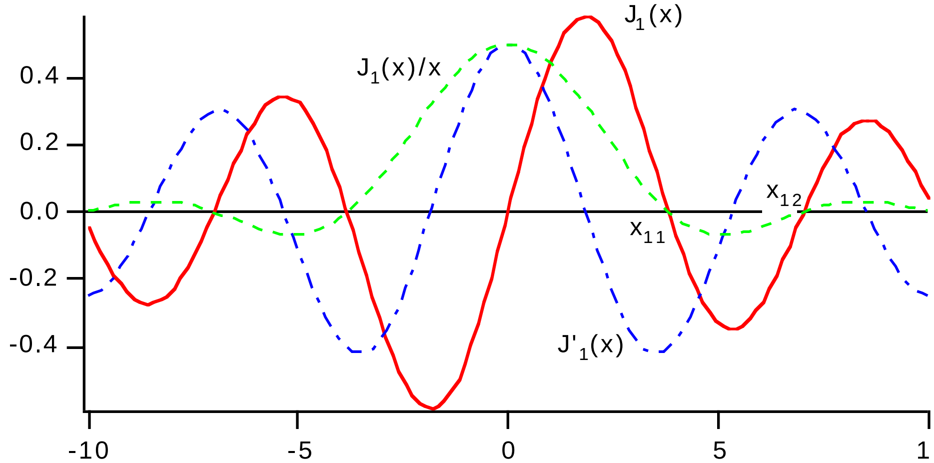
Dipole modes have one full period of variation around the azimuth. For TM modes this means there is no longitudinal field on axis and that the field strength grows linearly with radius close to the center, with opposite sign either side of the axis. This transverse gradient to the longitudinal field gives rise to a transverse voltage kick which is proportional to the beam current and the beam offset. This can be expressed through a **transverse impedance $Z_\perp$**:

$$Z_\perp[\Omega m^{-1}] = j\frac{-V_x}{I_b x_o}$$

where $I_b(0)x_o$ is the dipole moment of the beam. It can be shown that $Z_\perp$ is related to $Z_{||}$ by

$$Z_\perp[\Omega m^{-1}] = \frac{Z_{||}(r)}{kr^2}$$

where $Z_{||}(r)$ is the longitudinal impedance evaluated at radius r



For TM_{1ni} modes:

$E_z = E_o J_1(k_{1n}r)\cos(\phi)\cos(k_z z)$ where $k_{1n} = x_{1n}/a$ and $k_z = i\pi/\text{length}$ $(i \geq 0)$

$H_\phi = H_{\phi o} J_1'(k_{1n}r)\cos(\phi)\cos(k_z z)$ $\quad x_{11} = 3.383171$

$|H_r| = H_{ro}\frac{J_1'}{r}(k_{1n}r)\sin(\phi)\cos(k_z z)$ $\quad x_{12} = 7.01559$

$x_{13} = 10.17347$